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A Two-Grid Algorithm of Fully Discrete Galerkin Finite Element Methods for a Nonlinear Hyperbolic Equation

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Abstract. A two-grid finite element approximation is studied in the fully discrete scheme obtained by discretizing in both space and time for a nonlinear hyperbolic equation. The main idea of two-grid methods is to use a coarse-grid space (S_H) to produce a rough approximation for the solution of nonlinear hyperbolic problems and then use it as the initial guess on the fine-grid space (S_h). Error estimates are presented in H^1-norm, which show that two-grid methods can achieve the optimal convergence order as long as the two different girds satisfy $h = \mathcal{O}(H^2)$. With the proposed techniques, we can obtain the same accuracy as standard finite element methods, and also save lots of time in calculation. Theoretical analyses and numerical examples are presented to confirm the methods.

AMS subject classifications: 65N12, 65M60

Key words: Nonlinear hyperbolic equation, two-grid algorithm, finite element method, fully discrete, error estimate.

1. Introduction

In what follows, we study two-grid finite element methods for a nonlinear hyperbolic equation with Dirichlet boundary, defined by

$$\begin{cases} u_{tt} - \nabla \cdot (a(u)\nabla u) = f(u), & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with smooth boundary $\partial\Omega, u_0, u_1, a$ and f are known functions, and u_t, u_{tt} denotes $\partial u/\partial t, \partial^2 u/\partial t^2$, respectively. We assume that the problem (1.1) satisfies the following conditions:

i $a(u)$ is sufficiently smooth and bounded such that there exist constants a_i $(i = 0, 1)$ satisfying

$$0 < a_0 \leq a(u) \leq a_1. \tag{1.2}$$

ii $a(u), a_u(u)$ and $f(u)$ satisfy the Lipschitz continuous condition with respect to u_1, u_2,

$$|g(u_1) - g(u_2)| \leq L|u_1 - u_2|, \quad g = a, a_u, f, \tag{1.3}$$

where L is a positive constant, $a_u(u)$ denotes $\partial a(u)/\partial u$.

Nonlinear hyperbolic equations have the good application value in nonlinear vibration, flow mechanics and other practical problems. In decades, many researchers have focused on nonlinear hyperbolic equations in different numerical methods. For examples, Yuan and Wang [28,37] proved the stability and convergence of the finite element method for nonlinear hyperbolic equations. Eymard [14] studied an error estimate of the finite volume approximation for a nonlinear hyperbolic equation. Tadmor [25] studied local error estimates for discontinuous solutions of nonlinear hyperbolic equations. Li and Wei [20] derived optimal error estimates by semi-discrete and fully discrete finite element methods for nonlinear hyperbolic equations with nonlinear boundary condition. Chen and Huang [9] applied fully discrete mixed finite element methods to study second order nonlinear hyperbolic equations. Jiang and Chen [18] studied the application of mixed finite element methods for a strongly nonlinear second-order hyperbolic equation in divergence form. Chen *et al.* [10] derived error estimates of the full-discrete mixed finite element method for nonlinear hyperbolic problems. Liu [22,23] applied the finite element method to study nonlinear hyperbolic equations. Shi and Li [24] analyzed the superconvergence of finite element methods for nonlinear hyperbolic equations with nonlinear boundary condition. Lai and Yuan [19] studied the Galerkin alternating-direction method for a kind of three-dimensional nonlinear hyperbolic problems. Fedotov [15] analyzed nonconformal schemes of the finite-element method for nonlinear hyperbolic conservation laws. Wang and Guo [29] applied the nonconforming mixed finite element method to study nonlinear hyperbolic equations. Zhou *et al.* [38] studied an H^1-Galerkin expanded mixed finite element approximation of second-order nonlinear hyperbolic equations. Wang and Guo [30] applied a new approach to convergence analysis of linearized finite element methods for nonlinear hyperbolic equations.

On the other hand, two-grid finite element methods, based on two linear conforming finite element spaces S_H and S_h on one coarse grid H and one fine grid h, respectively, are firstly introduced by Xu [34,35] to solve nonsymmetric linear and nonlinear