A Two-Grid Algorithm of Fully Discrete
Galerkin Finite Element Methods for
a Nonlinear Hyperbolic Equation

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Abstract. A two-grid finite element approximation is studied in the fully discrete scheme obtained by discretizing in both space and time for a nonlinear hyperbolic equation. The main idea of two-grid methods is to use a coarse-grid space \((S_H)\) to produce a rough approximation for the solution of nonlinear hyperbolic problems and then use it as the initial guess on the fine-grid space \((S_h)\). Error estimates are presented in \(H^1\)-norm, which show that two-grid methods can achieve the optimal convergence order as long as the two different grids satisfy \(h = O(H^2)\). With the proposed techniques, we can obtain the same accuracy as standard finite element methods, and also save lots of time in calculation. Theoretical analyses and numerical examples are presented to confirm the methods.

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Key words: Nonlinear hyperbolic equation, two-grid algorithm, finite element method, fully discrete, error estimate.

1. Introduction

In what follows, we study two-grid finite element methods for a nonlinear hyperbolic equation with Dirichlet boundary, defined by

\[
\begin{align*}
&u_{tt} - \nabla \cdot (a(u)\nabla u) = f(u), & (x,t) \in \Omega \times (0,T], \\
&u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\
&u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \Omega,
\end{align*}
\]

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where $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with smooth boundary $\partial \Omega$, $u_0$, $u_1$, $a$ and $f$ are known functions, and $u_t, u_{tt}$ denotes $\partial u/\partial t, \partial^2 u/\partial t^2$, respectively. We assume that the problem (1.1) satisfies the following conditions:

i. $a(u)$ is sufficiently smooth and bounded such that there exist constants $a_i (i = 0, 1)$ satisfying

$$0 < a_0 \leq a(u) \leq a_1.$$  (1.2)

ii. $a(u), a_u(u)$ and $f(u)$ satisfy the Lipschitz continuous condition with respect to $u_1, u_2$,

$$|g(u_1) - g(u_2)| \leq L|u_1 - u_2|, \quad g = a, a_u, f,$$  (1.3)

where $L$ is a positive constant, $a_u(u)$ denotes $\partial a(u)/\partial u$.


On the other hand, two-grid finite element methods, based on two linear conforming finite element spaces $S_H$ and $S_h$ on one coarse grid $H$ and one fine grid $h$, respectively, are firstly introduced by Xu [34, 35] to solve nonsymmetric linear and nonlinear