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Effect of Flux Evaluation Methods on the Resolution and Robustness of the Two-Step Finite-Difference WENO Scheme

T. Kamiya[1], M. Asahara[2,*] and T. Nonomura[3]

[1] *Mechanical and Civil Engineering Division, Graduate School of Engineering, Gifu University, Yangido 1-1, Gifu-shi, Gifu, Japan*
[2] *Faculty of Engineering, Gifu University, Yangido 1-1, Gifu-shi, Gifu, Japan*
[3] *Department of Aerospace Engineering, Graduate School of Engineering, Tohoku University, Japan*

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Abstract. The resolution and the robustness of the weighted essentially non-oscillatory (WENO) scheme and two-step finite-difference WENO (TSFDWENO) schemes are compared by strictly using the same flux evaluation method and smoothness indicators. TSFDWENO schemes are defined to include a family of weighted compact nonlinear scheme (WCNS) and an alternative WENO scheme. Comparison results indicate that WCNS has a higher resolution than the WENO scheme, while the WENO scheme is more robust than WCNS. Additionally, various flux evaluation methods are combined with TSFDWENO schemes, and they are evaluated. Then, the effects of the flux evaluation methods on the resolution and robustness of the scheme are investigated, and the results show that the robustness and the resolution can be significantly altered by changing the flux evaluation method. This study reveals the advantage of being able to use various flux evaluation methods in the TSFDWENO scheme as well as the fair comparison of the WENO schemes and WCNS. On the other hand, these effects are marginalized when changing the interpolation and differencing method. Such knowledge can be important when selecting schemes for actual simulation and developing guidelines for scheme improvement.

Key words: High-order scheme, resolution, robustness, flux evaluation, TSFDWENO scheme.

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1. Introduction

Methods for the high-order accuracy calculation of compressible flow, where discontinuous surfaces exist, such as in shock waves or contact surfaces, include the weighted essentially non-oscillatory (WENO) [10] scheme and the weighted compact

*Corresponding author. *Email address:* `asahara@gifu-u.ac.jp` (M. Asahara)

nonlinear scheme (WCNS) [4, 20, 30], whereas the latter combines the WENO scheme with a compact scheme. Using these schemes instead of other schemes that are frequently used can reduce the computational cost while maintaining the desired resolution. Therefore, these schemes are widely prevalent and applied to various phenomena, such as detonation, acoustic flow, and two-phase flow. Many researchers have also conducted various evaluations and improvements on these schemes (e.g., [1, 3, 4, 6, 8, 10–12, 15, 17, 19–21, 30]). For example, Li *et al.* [15] proposed a high-order arbitrary Lagrangian-Eulerian finite difference WENO scheme for the Hamilton-Jacobi equations. Liu and Hu [17] proposed a block-structures adaptive mesh method coupled with a modified hybrid WENO scheme.

Compared with the WENO scheme, WCNS has three unique characteristics: it has high resolution [4, 30], various flow evaluations can be used [4], and freestream can be maintained on a wavy grid [19]. However, comparisons of the resolution of the WENO scheme and WCNS in previous studies were unfair because the flux evaluation method or the smoothness indicator differ. For example, Deng and Zhang [4] reported that the corner of the rarefaction wave calculated by WCNS is improved compared with that calculated by WENO scheme in the one-dimensional shock tube problem. However, the smoothness indicators of these schemes differ. Additionally, the flux evaluation methods of the WENO scheme are Lax-Friedrichs flux splitting (LF) or the Roe flux splitting (RF). In contrast flux difference splitting (FDS) is adopted in WCNS as the flux evaluation method. To our knowledge no comprehensive comparison of resolution and robustness has been performed between the original WENO scheme and WCNS.

Nonomura and Fuji [21] and Deng *et al.* [3] recently proposed new linear formulations of WCNS that use cell-node values in addition to cell-edge values. Nonomura and Fuji [21] proposed a robust WCNS (RWCNS) with greater robustness than original WCNS. Deng *et al.* [3] proposed hybrid cell-edge and cell-node WCNS (HWCNS) with a narrower stencil width, yet with the same accuracy as the original WCNS. HWCNS has a constant parameter that controls dissipation. Asahara *et al.* [1] recently indicated that WCNS, RWCNS, and HWCNS derived from the WENO scheme are constructed by the following two steps: 1) high-order nonlinear interpolation and flux evaluation and 2) high-order linear difference of the numerical flux. They named these methods two-step finite-difference WENO (TSFDWENO) schemes. Moreover, their study exhibited that the alternative WENO scheme [11] is equivalent to a special case of HWCNS.

While the WENO scheme interpolates the flux, WCNS makes it possible to perform variable interpolation in addition to flux interpolation. In previous research, Zhang *et al.* [30] implemented the scheme on a flux interpolation version, which similar to how the WENO scheme operates, and Deng *et al.* [4] and Nonomura *et al.* [20] implemented a variable interpolation version of WCNS. Hence, many flux evaluation methods can be used in WCNS: FDS of an approximate Riemann solver [22], flux vector splitting (FVS) [27], advection upwind splitting method (AUSM) [16], and the Harten-Lax-Leer (HLL) system [7]. In particular, various flux evaluation methods implementing modifications to the AUSM and HLL system were proposed. While the WENO scheme interpolates the flux, as mentioned above. Therefore, the WENO scheme