Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2020-0016

Vol. **13**, No. 4, pp. 1098-1122
November 2020

A New Iteration and Preconditioning Method for Elliptic PDE-Constrained Optimization Problems

Owe Axelsson[1] and Davod Khojasteh Salkuyeh[2,3,*]

[1] *The Czech Academy of Sciences, Institute of Geonics, Ostrava, Czech Republic*
[2] *Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[3] *Center of Excellence for Mathematical Modelling, Optimization and Combinational Computing (MMOCC), University of Guilan, Rasht, Iran*

Received 27 January 2020; Accepted (in revised version) 28 April 2020

Abstract. Optimal control problems constrained by a partial differential equation (PDE) arise in various important applications, such as in engineering and natural sciences. Normally the problems are of very large scale, so iterative solution methods must be used. Thereby the choice of an iteration method in conjunction with an efficient preconditioner is essential. In this paper, we consider a new iteration method and a new preconditioning technique for an elliptic PDE-constrained optimal control problem with a distributed control function. Some earlier used iteration methods and preconditioners in the literature are compared, both analytically and numerically with the new iteration method and the preconditioner.

AMS subject classifications: 49M25, 49K20, 65F10, 65F50

Key words: Preconditioner, hybrid, PRESB, GMRES, PDE-constrained optimization, optimization.

1. Introduction

We are concerned with the following linear-quadratic elliptic distributed optimal control problems

$$\min_{u,f} \quad \frac{1}{2}\|u-u_*\|^2_{L^2(\Omega)} + \beta\|f\|^2_{L^2(\Omega)}, \tag{1.1}$$

$$\text{s.t.} \quad \mathcal{L}u = f \quad \text{in } \Omega, \tag{1.2}$$

$$u = g_1 \quad \text{on } \partial\Omega_1, \quad \frac{\partial u}{\partial n} = g_2 \quad \text{on } \partial\Omega_2, \tag{1.3}$$

*Corresponding author. *Email addresses:* `khojasteh@guilan.ac.ir` (D.K. Salkuyeh), `owe.axelsson@it.uu.se` (O. Axelsson)

where $\partial\Omega$ is the boundary of domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\partial\Omega := \partial\Omega_1 \cup \partial\Omega_2$ and $\partial\Omega_1 \cap \partial\Omega_2 = \phi$. Here u is the state variable to be computed and f is the control function, distributed on the whole domain of definition, Ω. $\mathcal{L}$ is a given elliptic operator. For notational simplicity we take $\mathcal{L} \equiv -\Delta$. We also assume that $\partial\Omega_1$ is not empty. The problem is ill-posed if $\beta = 0$, so $\beta > 0$ is a regularization parameter and u_* is a given function representing the desired state. Lions in [15] introduced the above class of problems, which consists of minimizing a cost functional subject to a partial differential equation (PDE) problem in Ω. Note that if the function u_* is sufficiently smooth, i.e., belongs to $C^2(\overline{\Omega})$, and satisfies the given boundary conditions, then the control function can be computed as $f = -\Delta u_*$. We do not consider such special cases here. In practical problems the observation is often restricted to a subdomain of Ω or to its boundary. In this case, the first term in (1.1) is replaced by $\frac{1}{2}\|u - u_*\|^2_{L^2(\partial\Omega)}$. For further discussions about optimal control problems with limited observation domains, see, e.g., [16] . In Section 4 we comment also on a modification of our preconditioning method which can handle such a case also.

There are two choices for solving the problem: discretize-then-optimize and optimize -then-discretize. Using the discretize-then-optimize approach Rees *et al.* in [23], transformed the problem to a saddle point system. They applied the Galerkin finite-element method to the weak formulation of Eqs. (1.1)-(1.3) and obtained the finite-dimensional optimization problem

$$\begin{cases} \min_{\mathbf{u},\mathbf{f}} & \frac{1}{2}\mathbf{u}^T M\mathbf{u} - \mathbf{u}^T\mathbf{b} + \frac{1}{2}\|\mathbf{u}_*\|_2^2 + \beta\mathbf{f}^T M\mathbf{f}, \\ \text{s.t.} & K\mathbf{u} = M\mathbf{f} + \mathbf{d}, \end{cases} \tag{1.4}$$

where $M \in \mathbb{R}^{n\times n}$ is the mass matrix and $K \in \mathbb{R}^{n\times n}$ is the stiffness matrix (here, the discrete Laplacian), the entries of the vector $\mathbf{d} \in \mathbb{R}^n$ come from the boundary values of the discrete solution, and $\mathbf{b} = Pu^* \in \mathbb{R}^n$, where P is the projection matrix on the finite element space. Applying the Lagrange multiplier technique to the minimization problem (1.4) results in the following system of linear equations

$$\mathcal{A}\mathbf{x} = \begin{pmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} = \mathbf{g}, \tag{1.5}$$

where $\lambda \in \mathbb{R}^n$ is a vector of Lagrange multipliers, i.e. the adjoint variable to u.

The main purpose of the paper is to present a new method and to compare methods to solve the system. Both of the matrices M and K, and hence also the matrix $\mathcal{A}$, are sparse. Therefore, we shall use iterative solution methods for which the choice of an efficient preconditioner is essential. Further, the matrix M is symmetric positive definite. Therefore, in general the matrix $\mathcal{A}$ is symmetric and indefinite. Hence, the MINRES method of Paige and Saunders [21] can be applied for solving the system (1.5). To improve the convergence of the MINRES method for solving (1.5), several preconditioners have been presented in the literature. For an early reference to the use of preconditioned Schur-complement matrices, see [1]. In the next section, we give