Numer. Math. Theor. Meth. Appl. Vol. **14**, No. 1, pp. 1-30
doi: 10.4208/nmtma.OA-2020-0034 February 2021

Mean-Square Approximation of Navier-Stokes Equations with Additive Noise in Vorticity-Velocity Formulation

G.N. Milstein[1] and M.V. Tretyakov[2,*]

[1] *Ural Federal University, Ekaterinburg, Russia*
[2] *School of Mathematical Sciences, University of Nottingham, Nottingham, UK*

Received 24 February 2020; Accepted (in revised version) 27 August 2020

Abstract. We consider a time discretization of incompressible Navier-Stokes equations with spatial periodic boundary conditions and additive noise in the vorticity-velocity formulation. The approximation is based on freezing the velocity on time subintervals resulting in a linear stochastic parabolic equation for vorticity. At each time step, the velocity is expressed via vorticity using a formula corresponding to the Biot-Savart-type law. We prove the first mean-square convergence order of the vorticity approximation.

AMS subject classifications: 65C30, 60H15, 60H35, 35Q30

Key words: Navier-Stokes equations, vorticity, numerical method, stochastic partial differential equations, mean-square convergence.

---

1. Introduction

Navier-Stokes equations (NSE), both deterministic and stochastic, are important for a number of applications and, consequently, development and analysis of numerical methods for simulation of NSE are of significant interest. The theory and applications of stochastic NSE (SNSE) can be found, e.g. in [10, 16, 19]. The literature on numerics for deterministic NSE is extensive [12, 27, 29] (see also references therein) while the literature on numerics for SNSE is still rather sparse, let us mention [2–6, 8, 9, 25]. In comparison with the previous works [2–6, 9, 25] where strong approximation of SNSE in velocity formulation was considered, we here deal with the vorticity-velocity formulation, and, as far as we know, this is the first work in this direction. In [8] a similar setting to ours was used but in the context of weak approximation.

In this paper we consider two-dimensional incompressible NSE in the vorticity-velocity formulation with periodic boundary conditions and additive noise (see, e.g.

*Corresponding author. *Email address:* `Michael.Tretyakov@nottingham.ac.uk` (M.V. Tretyakov)

[13]). In majority of papers on numerical approximation of SNSE [2–6, 9] the case of multiplicative noise is considered. The NSE with additive noise deserves a special attention due to its interesting properties [13, 16, 18]. Also, we know [23] that mean-square order of convergence of numerical methods for ordinary stochastic differential equations (SDEs) with additive noise is typically higher than with multiplicative one, which led to special consideration of SDEs with additive noise in stochastic numerics. Here we follow this path in the case of SNSE.

We propose and study time discretization of SNSE in the vorticity-velocity formulation, which is based on freezing the velocity at every time step. Consequently, at every step we just need to solve a linear parabolic stochastic PDE, which is a much simpler object than SNSE. To compute the velocity, we express it via the vorticity, i.e. via a periodic version of Biot-Savart's law (see, e.g., [15]). We prove properties, including first-order mean-square convergence, of the suggested approximation. Since we work in the vorticity-velocity formulation and aimed at reaching a higher order of mean-square convergence for the method considered, we require higher spatial smoothness of the velocity in our proofs than e.g. in [3] where mean-square convergence of fully and semi-implicit Euler schemes from [6] for SNSE with multiplicative and additive noise in the velocity formulation was considered. In the case of an additive noise, the authors of [3] proved mean-square convergence with a polynomial rate (up to $\frac{1}{4}$) in the time mesh. To obtain this result, they used some exponential moments bounds of the SNSE solutions analogous to the ones from [13]. Here to prove first-order mean-square convergence of our new method for SNSE in the vorticity-velocity formulation, we exploit some exponential moments bounds for vorticity.

The benefit of working within the vorticity-velocity formulation is that we do not need to deal with the divergence free condition imposed on the velocity. We remark that, as it is usual in numerical analysis, the suggested approximation can be used in practice even if the regularity conditions required for the proofs are not satisfied. In this paper our main objective is to propose a new approximation for SNSE and to prove its highest possible mean-square order of convergence under some regularity assumptions on the solution. Alternatively, one can pose the question of establishing a convergence of the proposed method under prescribed low regularity conditions, this is a topic for a possible future work (see also Remark 5.1 at the end of the paper).

The paper is organised as follows. In Section 2, after introducing the SNSE in velocity formulation, we recall function spaces required. Then we write the SNSE in the vorticity-velocity formulation (Section 2.2) and prove two auxiliary lemmas (Section 2.3) about its solution. These lemmas are of independent interest. The lemmas are later used in proving properties of the proposed approximation. We consider a one-step approximation of vorticity and its properties in Section 3. We introduce the numerical method for vorticity and prove boundedness of its moments in Section 4. We note that to turn the proposed method into a numerical algorithm one needs to approximate a linear SPDE at every step. Various approaches can be used for this purpose as we discuss in the end of the paper, and their detailed error analysis is a subject of a future work. We prove first-order mean-square convergence of the method in Section 5. Ideas