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A New Ensemble HDG Method for Parameterized Convection Diffusion PDEs

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Abstract. A new second order time stepping ensemble hybridizable discontinuous Galerkin method for parameterized convection diffusion PDEs with various initial and boundary conditions, body forces, and time depending coefficients is developed. For ensemble solutions in $L^\infty(0,T;L^2(\Omega))$, a superconvergent rate with respect to the freedom degree of the globally coupled unknowns for all the polynomials of degree $k\ge 0$ is established. The results of numerical experiments are consistent with the theoretical findings.

AMS subject classifications: 65M60

Key words: HDG method, ensemble methods, parameterized convection diffusion PDEs, numerical analysis.

1. Introduction

In this work, we propose a new second order time stepping ensemble hybridizable discontinuous Galerkin (HDG) method to efficiently simulate a group of parameterized convection diffusion equations on a Lipschitz polyhedral domain $\Omega\subset\mathbb{R}^d$ $(d\ge 2)$. For $j=1,\dots,J$, find $(\boldsymbol{q}_j,u_j)$ satisfying

$$\begin{aligned}
&c_j\boldsymbol{q}_j+\nabla u_j=0 && \text{in } \Omega\times(0,T],\\
&\partial_t u_j+\nabla\cdot\boldsymbol{q}_j+\boldsymbol{\beta}_j\cdot\nabla u_j=f_j && \text{in } \Omega\times(0,T],\\
&u_j=g_j && \text{on } \partial\Omega\times(0,T],\\
&u_j(\cdot,0)=u_j^0 && \text{in } \Omega,
\end{aligned}\tag{1.1}$$

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where

$$c_j := c_j(\boldsymbol{x}, t), \quad f_j := f_j(\boldsymbol{x}, t), \quad g_j := g_j(\boldsymbol{x}, t), \quad \boldsymbol{\beta}_j := \boldsymbol{\beta}_j(\boldsymbol{x}, t), \quad u_j^0 := u_j^0(\boldsymbol{x})$$

are given functions.

For many computational applications in real life, one needs to solve a group of PDEs with different input conditions, like the applications in petroleum engineering, which need to predict the transport properties of rock core-sample in centimeter scale. We need to capture the flow capacity of every single nanopore with different inputs, and the porous media of shale core-sample is composed of more than 10^6 pores. However, to efficiently simulate a group of PDEs with different inputs is a great challenge.

A first order time stepping ensemble method was proposed by [16] to study a set of J solutions of the Navier-Stokes equations with different initial conditions and forcing terms. The J solutions are computed simultaneously by solving a linear system with one common coefficient matrix and multiple RHS vectors. This leads to a great computational efficiency in linear solvers when either the LU factorization (for small-scale systems) or a block iterative algorithm (for large-scale systems) is used. Later, a second order time stepping ensemble algorithm was designed in [14]. Recently, a new ensemble method was proposed to treat the PDEs which have different coefficients [11, 12]. The ensemble method has been applied to many different models [8–10, 15, 17, 18, 20]. It is worthwhile to mention that the previous works only obtained a *suboptimal* $L^\infty(0, T; L^2(\Omega))$ convergence rate for the ensemble solutions.

More recently, we proposed a first order time stepping ensemble hybridizable discontinuous Galerkin (HDG) method in [3] to study a group of convection diffusion PDEs with different initial conditions, boundary conditions, body forces and coefficients. We obtained an *optimal* $L^\infty(0, T; L^2(\Omega))$ convergence rate for the solutions on a simplex mesh, and we obtained a $L^2(0, T; L^2(\Omega))$ superconvergent rate if the polynomials of degree $k \geq 1$ and the coefficients of the PDEs are independent of time. This ensemble HDG method uses polynomials of degree k for all variables, i.e., the flux variables $\boldsymbol{q}_j$ and the scalar variables u_j.

In this work, we devise a new second order time stepping ensemble HDG method for a group of convection diffusion PDEs. We use polynomials degree k to approximate the fluxes and the numerical traces, and use polynomials degree $k + 1$ to approximate the scale variable. This method was proposed by [19] and later analyzed by [21] for a single steady elliptic PDEs, they obtained a superconvergent rate for the scalar variable for all $k \geq 0$. This HDG method has been extended to study the PDEs with a convection term by [23, 24].

In this paper, we first restore the superconvergence for $k = 0$ by modifying the stabilization function in [23]. Next, we show that the new ensemble HDG method can obtain a $L^\infty(0, T; L^2(\Omega))$ superconvergent rate for all $k \geq 0$ on a general polyhedron mesh and without assume the coefficients are independent of time. It is worth mentioning that this new ensemble HDG method keep the advantages of the ensemble methods, i.e., all realizations share one common coefficient matrix and multiple RHS