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# Convergence and Stability of the Truncated Euler-Maruyama Method for Stochastic Differential Equations with Piecewise Continuous Arguments

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**Abstract.** In this paper, we develop the truncated Euler-Maruyama (EM) method for stochastic differential equations with piecewise continuous arguments (SDEPCAs), and consider the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. The order of convergence is obtained. Moreover, we show that the truncated EM method can preserve the exponential mean square stability of SDEPCAs. Numerical examples are provided to support our conclusions.

**AMS subject classifications**: 65C30
**Key words**: Stochastic differential equations with piecewise continuous argument, local Lipschitz condition, Khasminskii-type condition, truncated Euler-Maruyama method, convergence and stability.

## 1. Introduction

Differential equations with piecewise continuous arguments (EPCAs) have the following form

$$dx(t) = a\big(x(t), x(u(t))\big)dt, \tag{1.1}$$

where the argument $u(t)$ has the intervals of constancy, such as $u(t) = [t], [t-n], t-n[t]$. EPCAs have plenty of useful applications in the stabilization of hybrid control systems with feedback discrete controller [9,17,25,28]. Note that $[t]$ is a discontinuous function and the solutions are determined by a finite set of initial data, rather than an initial function. Moreover, EPCAs represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations [27].

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However, [21] pointed out that substantial theories of functional differential equations did not exist for differential equations with piecewise constant or piecewise continuous arguments. Therefore, it is necessary to extend the theory of delay differential or functional differential equations with continuous arguments to equations with discontinuous arguments.

Actually, the effects of noise are not negligible for some systems [22]. Therefore, the stochastic differential equations with piecewise continuous arguments (SDEPCAs) are used to describe many practical problems. It has been widely used in the neural networks, control theory and so on [11, 23, 28]. Since there is, in general, no explicit solutions to SDEPCAs, numerical solutions are required in practice. The convergence and stability of numerical methods are the central properties in numerical analysis, and it is well studied for SDEPCAs under global Lipschitz and linear growth conditions, such as [20]. However, many systems do not satisfy the global Lipschitz or the linear growth conditions. Song *et al.* [24] investigated the convergence of the tamed Euler method for SDEPCAs under non-global Lipschitz continuous coefficients.

According to [7], the classical explicit Euler method diverges when the coefficients of stochastic differential equations (SDEs) are super-linearly growing. To overcome this difficulty, some scholars turn to use implicit Euler methods, such as the semi-implicit Euler-Maruyama method, the backward Euler-Maruyama method and the split-step theta (SST) method [2, 4, 6, 19]. However, compared with explicit methods, implicit methods require more additional computational efforts and cost much more time. Therefore, many scholars pay attention to find an explicit method which will converge in the strong sense when the coefficients of SDEs satisfy the super-linear growth condition. At present, there are some effective methods, such as the stopped Euler method [12], the tamed Euler method [1, 8, 26], the truncated Euler-Maruyama method [5, 16, 18], the partially truncated Euler-Maruyama method [3] and the modified truncated Euler-Maruyama method [10]. In this paper, we develop the truncated Euler-Maruyama (EM) method for SDEPCAs, and consider the strong convergence theory under the local Lipschitz condition plus the Khasminskii-type condition. The order of convergence is obtained.

Guo *et al.* [3] investigated the mean square exponential stability of the partially truncated Euler-Maruyama method for SDEs. Later, Lan and Xia [10] gave the exponential stability of the modified truncated Euler-Maruyama method for SDEs. Hu *et al.* [5] obtained the asymptotic stability of the truncated Euler-Maruyama method for SDEs. In this paper, we show that the truncated Euler-Maruyama method preserves the exponential mean square stability of the SDEPCAs.

The rest of this article is organized as follows. Section 2 introduces some basic assumptions, definitions and properties of the exact solution. We construct the truncated Euler-Maruyama method in Section 3. The $p$-th moment boundedness of the numerical solutions and the convergence theorem are presented in Section 4. Section 5 obtains the convergence order and Section 6 shows the exponential stabilities of both the SDEPCAs and the numerical solutions. Numerical simulations are provided to verify the analytical theory in Section 7.