Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2020-0065

Vol. **14**, No. 1, pp. 31-46
February 2021

On Poincaré-Friedrichs Type Inequalities for the Broken Sobolev Space $\mathbf{W^{2,1}}$

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Received 17 April 2020; Accepted (in revised version) 5 August 2020

Abstract. We are concerned with the derivation of Poincaré-Friedrichs type inequalities in the broken Sobolev space $W^{2,1}(\Omega;\mathcal{T}_h)$ with respect to a geometrically conforming, simplicial triagulation $\mathcal{T}_h$ of a bounded Lipschitz domain Ω in $\mathbb{R}^d, d \in \mathbb{N}$. Such inequalities are of interest in the numerical analysis of nonconforming finite element discretizations such as C^0 Discontinuous Galerkin (C^0DG) approximations of minimization problems in the Sobolev space $W^{2,1}(\Omega)$, or more generally, in the Banach space $BV^2(\Omega)$ of functions of bounded second order total variation. As an application, we consider a C^0DG approximation of a minimization problem in $BV^2(\Omega)$ which is useful for texture analysis and management in image restoration.

AMS subject classifications: 65K10, 65N30, 68U10

Key words: Poincaré-Friedrichs inequalities, broken Sobolev spaces, C^0 Discontinuous Galerkin approximation, image processing.

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1. Introduction

Poincaré-Friedrichs type inequalities for broken Sobolev spaces play an important role in the numerical analysis of nonconforming finite element discretizations of minimization problems in Sobolev spaces and associated partial differential equations (cf., e.g., [6–8, 12, 13]). In this paper, given a bounded Lipschitz domain Ω in $\mathbb{R}^d, d \in \mathbb{N}$, we derive such inequalities for the broken Sobolev space $W^{2,1}(\Omega;\mathcal{T}_h)$ with respect to a geometrically conforming, simplicial triangulation $\mathcal{T}_h$ of Ω. They are based on Poincaré-Friedrichs type inequalities for the space $BV^2(\Omega)$ of functions of bounded second order total variation [3, 4]. As an application, we consider a C^0 Discontinuous Galerkin (C^0DG) approximation of a minimization problem in $BV^2(\Omega)$ which is used in image processing for texture analysis and management.

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The paper is organized as follows: In Section 2, we introduce the Banach spaces $BV(\Omega)$ and $BV^2(\Omega)$ of functions of bounded first and second order total variation and recall some of its properties that pertain to the derivation of the Poincaré-Friedrichs type inequalities, whereas Section 3 is devoted to Poincaré-Friedrichs type inequalities in $BV^2(\Omega)$. In Section 4, we define the broken Sobolev space $W^{2,1}(\Omega;\mathcal{T}_h)$ in terms of a broken Hessian involving a recovery operator from the broken Sobolev space $W^{2,1}(\Omega;\mathcal{T}_h)$ into the linear space of $d\times d$ matrices with element-wise polynomial entries. We prove the boundedness of the recovery operator in the L^1 norm (Theorem 4.1). In Section 5, we first show that for functions in $W^{2,1}(\Omega;\mathcal{T}_h)$ the second order total variation can be bounded from above by the $W^{2,1}(\Omega;\mathcal{T}_h)$ seminorm and obtain a compactness result in the sense that bounded sequences in $W^{2,1}(\Omega;\mathcal{T}_h)$ contain a subsequence converging weakly* in $BV^2(\Omega)$ (Theorem 5.1). We then derive two Poincaré-Friedrichs type inequalities for the broken Sobolev space $W^{2,1}(\Omega;\mathcal{T}_h)$ (Theorem 5.2). Finally, in Section 6 we consider the C^0DG approximation of a minimization problem in $BV^2(\Omega)$ which can be applied to texture analysis and management in image restoration [4, 5] and prove that the sequence of C^0DG approximations in $W^{2,1}(\Omega;\mathcal{T}_h)$ contains a subsequence converging weakly* in $BV^2(\Omega)$ to a solution of the original minimization problem (Theorem 6.1).

2. Functions of bounded first and second order total variation

For a bounded Lipschitz domain $\Omega\subset\mathbb{R}^d, d\in\mathbb{N}$, with boundary $\Gamma=\partial\Omega$ we refer to $C_0^m(\Omega), 0\le m\le\infty$, as the Banach space of m-times continuously differentiable scalar functions with compact support in Ω. Likewise, $C_0^m(\Omega;\mathbb{R}^d), 0\le m\le\infty$, stands for the Banach space of m-times continuously differentiable vector-valued functions with compact support in Ω and $C_0^m(\Omega;\mathbb{R}^{d\times d}), 0\le m\le\infty$, for the Banach space of m-times continuously differentiable matrix-valued functions with compact support in Ω.

Moreover, we will use standard notation from Lebesgue and Sobolev space theory [14]. In particular, for Lipschitz subsets $D\subseteq\bar{\Omega}$ and $1\le p\le\infty$ we denote the L^p-norm by $\|\cdot|_{L^p(D)}$. We further refer to $W^{m,p}(D), m\in\mathbb{N}$, as the Sobolev spaces with norm $\|\cdot\|_{W^{m,p}(D)}$, and seminorm $|\cdot|_{W^{m,p}(D)}$, and to $W^{m-\frac{1}{p},p}(\Gamma'), \Gamma'\subset\partial D$, as the associated trace spaces. $W_0^{m,p}(D)$ stands for the closure of $C_0^\infty(D)$ in the $W^{m,p}$-norm. Sobolev spaces $W^{s,p}(D)$ with broken index $s\in\mathbb{R}_+$ are defined by interpolation. In case $p=2$ we will write $H^m(D)$ instead of $W^{m,2}(D)$. The spaces $L^2(D)$ and $H^m(D)$ are Hilbert spaces with inner products denoted by $(\cdot,\cdot)_{L^2(D)}$ and $(\cdot,\cdot)_{H^m(D)}$. Further, $H^{-m}(D)$ refers to the dual space of $H_0^m(D)$ with $\langle\cdot,\cdot\rangle_{m,D}$ denoting the dual product.

The spaces $L^p(D;\mathbb{R}^d), 1\le p\le\infty$, stand for the Banach spaces of vector-valued functions $\underline{\mathbf{q}}=(q_1,\ldots,q_d)^T$ with norm $\|\underline{\mathbf{q}}\|_{L^p(D;\mathbb{R}^d)}:=(\int_D|\underline{\mathbf{q}}|^p\,dx)^{\frac{1}{p}}$ for $1\le p<\infty$, where $|\underline{\mathbf{q}}|:=(\underline{\mathbf{q}}\cdot\underline{\mathbf{q}})^{\frac{1}{2}}$ and $\underline{\mathbf{p}}\cdot\underline{\mathbf{q}}=\sum_{i=1}^d p_iq_i$, and $\|\underline{\mathbf{q}}\|_{L^\infty(D;\mathbb{R}^d)}:=\max_{1\le i\le d}\|q_i\|_{L^\infty(D)}$ for $p=\infty$. Likewise, we refer to $L^p(\Omega;\mathbb{R}^{d\times d}), 1\le p\le\infty$, as the Banach spaces of matrix-valued functions $\underline{\underline{\mathbf{q}}}=(q_{ij})_{i,j=1}^d$ with norm $\|\underline{\underline{\mathbf{q}}}\|_{L^p(D;\mathbb{R}^{d\times d})}:=(\int_D|\underline{\underline{\mathbf{q}}}|^p\,dx)^{\frac{1}{p}}$ for