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Numerical Analysis of a High-Order Scheme for Nonlinear Fractional Differential Equations with Uniform Accuracy

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Abstract. We introduce a high-order numerical scheme for fractional ordinary differential equations with the Caputo derivative. The method is developed by dividing the domain into a number of subintervals, and applying the quadratic interpolation on each subinterval. The method is shown to be unconditionally stable, and for general nonlinear equations, the uniform sharp numerical order $3-\nu$ can be rigorously proven for sufficiently smooth solutions at all time steps. The proof provides a general guide for proving the sharp order for higher-order schemes in the nonlinear case. Some numerical examples are given to validate our theoretical results.

AMS subject classifications: 65M06

Key words: Caputo derivative, fractional ordinary differential equations, high-order numerical scheme, stability and convergence analysis.

1. Introduction

In the past decades, fractional differential equations have been studied extensively by many researchers, due to its success in describing some physical phenomena and chemical processes more accurately than integer order differential equations [18, 30, 33, 34]. Like most classical differential equations, the exact solutions of fractional order differential equations are usually not available to us. Even if analytical solutions can be found, they usually appear in the form of series and are difficult to evaluate. Therefore, the numerical study of fractional differential equations has also inspired a number of excellent research works such as [6, 7, 12, 13, 15, 21, 29, 41].

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In this work, we are interested in the following initial value problem: For some $\nu \in (0,1)$, we would like to find $y(x)$ such that

$$ {}_0D_x^{\nu} y(x) = f(x, y(x)), \quad 0 < x \le T \tag{1.1} $$

subject to the initial condition $y(0) = y_0$. In (1.1), the operator ${}_0D_x^{\nu}$ is the Caputo derivative, defined by

$$ {}_0D_x^{\nu} y(x) = \int_0^x \omega_{1-\nu}(x-s) y'(s) ds, \tag{1.2} $$

where $\omega_{1-\nu}$ is defined by

$$ \omega_{1-\nu}(x) = \frac{x^{-\nu}}{\Gamma(1-\nu)} \tag{1.3} $$

with $\Gamma(\cdot)$ being Euler's gamma function. The function $\omega_{1-\nu}(x)$ acts as the convolutional kernel, which satisfies

$$ \int_s^t \omega_{\nu}(t-\mu)\omega_{1-\nu}(\mu - s) d\mu = \omega_1(t-s) = 1, \quad \forall\, 0 < s < t < +\infty. \tag{1.4} $$

The numerical method for this equation has been extensively studied in the context of linear partial differential equations. For example, the L1-type schemes based on piecewise linear interpolation have been studied in [5, 10, 27], where the numerical order is $2-\nu$. In [1], the second-order L2-1_σ method is proposed by quadratic interpolation. To achieve the sharp order $3-\nu$ for smooth solutions, one can use the L1-2 method proposed in [11], which is also based on the quadratic interpolation, or the method based on Taylor expansion as introduced in [20]. For this numerical order, fast numerical schemes to discretize the Caputo derivative is proposed in [39]. Generalization to $(r+1-\nu)$-th order schemes have been studied in [3, 23] by Lagrange interpolation. A common problem in these methods is that the theoretical order of the solution at the first time step can only achieve $2-\nu$, as is shown in the numerical analysis in [20]. Such a problem is also mentioned in [24], where the author uses a finer grid near the initial value to maintain the numerical accuracy. In [32], it is found that the size of the finer grid should be proportional to $\Delta t^{2-\nu}$, which may cause significant additional computational cost especially when ν is small. Moreover, it is pointed out in [16] that the realistic solution u is usually nonsmooth at $t = 0$, and the initial layer can also cause the reduction of numerical order. To solve this issue, graded meshes have been introduced to restore the numerical order [5, 36]. In this case, the size of the finer grid could be even smaller if a high-order scheme is needed. Therefore, we are motivated to find a scheme that does not require a finer mesh for the first time step. For simplicity, in this paper we restrict ourselves to the case of a uniform mesh, and assume the smoothness of the solution. Other related works include, but not limited to [2, 8, 38].

In principle, these methods can be directly generalized to nonlinear problems. However, the analysis of convergence order on such methods for nonlinear problems is less seen in the literature. In [4], the authors converted the Caputo fractional derivative