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Discrete Approximation Scheme in Distributionally Robust Optimization

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Abstract. Discrete approximation, which has been the prevailing scheme in stochastic programming in the past decade, has been extended to distributionally robust optimization (DRO) recently. In this paper, we conduct rigorous quantitative stability analysis of discrete approximation schemes for DRO, which measures the approximation error in terms of discretization sample size. For the ambiguity set defined through equality and inequality moment conditions, we quantify the discrepancy between the discretized ambiguity sets and the original set with respect to the Wasserstein metric. To establish the quantitative convergence, we develop a Hoffman error bound theory with Hoffman constant calculation criteria in a infinite dimensional space, which can be regarded as a byproduct of independent interest. For the ambiguity set defined by Wasserstein ball and moment conditions combined with Wasserstein ball, we present similar quantitative stability analysis by taking full advantage of the convex property inherently admitted by Wasserstein metric. Efficient numerical methods for specifically solving discrete approximation DRO problems with thousands of samples are also designed. In particular, we reformulate different types of discrete approximation problems into a class of saddle point problems with completely separable structures. The stochastic primal-dual hybrid gradient (PDHG) algorithm where in each iteration we update a random subset of the sampled variables is then amenable as a solution method for the reformulated saddle point problems. Some preliminary numerical tests are reported.

AMS subject classifications: 90C15, 90C46, 90C47

Key words: Quantitative stability analysis, Hoffman lemma, discrete approximation method, distributionally robust optimization, stochastic primal-dual hybrid gradient.

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1. Introduction

Making an optimal decision under uncertain conditions is typically a challenging task in decision analysis. The quality of such decisions relies heavily on the information concerning the underlying uncertainties. Stochastic programming (SP) is a powerful tool if the uncertainty distributions are completely known. The history of SP can be traced back to the middle of the last century. So far, a variety of SP models have been proposed to handle the presence of random data in optimization problems. Prevailing examples include chance-constrained models, two-and multi-stage models, and models involving risk measures. For recent developments on SP, we refer the readers to a monograph [39] and references therein.

Unfortunately, in most real-life applications such as signal processing of mobile ad hoc networks, the number of samples collected is relatively small. Evaluating the exact probability of safe operation is somewhat challenging. One remedy for this difficulty is to adopt a distributionally robust approach. That is, constructing an ambiguity set of distributions with historical data, computer simulations or subjective judgements that contains the true distribution with certain confidence. Thus we may choose an optimal decision on the basis of the worst-case distribution over the ambiguity set. For example, some samples can be obtained, it may be more reliable to estimate the moment information than to evaluate the exact probability. This type of robust optimization framework can be traced back to the earlier work by Scarf [40]. It has been thoroughly investigated through research, see, e.g., Žáčková [51], Dupačová [14], and Shapiro and Ahmed [42]. Over the past few years, it has gained substantial popularity through further contributions by Bertsimas and Popescu [5], Betsimas *et al*. [4], Delage and Ye [11], Goldfarb and Iyengar [19], Mehrotra and Papp [27], Pflug *et al*. [31], Wiesemann *et al*. [46, 47].

One important issue concerning distributionally robust optimization (DRO) is the design of numerical solvers. Unlike robust optimization problems, DRO problems usually contain functional variables. Designing implementable and efficient numerical schemes for solving DRO becomes extraordinarily challenging. Most studies of DRO have focused on a dual approach. In summary, the technical framework of such approach has three steps:

- Consider the Lagrange dual of the inner max problem.
- Reformulate the min-max problem as a min-min (combining the min-min by min) problem with semi-infinite constraints.
- Recast the semi-infinite constraints as a linear semi-definite constraint by S-Lemma or dual method again.

In particular, Wiesemann *et al*. [48] provides a unified framework of the semi-definite programming (SDP) reformulation for DRO problems where the ambiguity set is constructed through some probabilistic and moment constraints.