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An Improved Linearity-Preserving Cell-Centered Scheme for Nonlinear Diffusion Problems on General Meshes

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Abstract. In this paper, we suggest a new vertex interpolation algorithm to improve an existing cell-centered finite volume scheme for nonlinear diffusion problems on general meshes. The new vertex interpolation algorithm is derived by applying a special limit procedure to the well-known MPFA-O method. Since the MPFA-O method for 3D cases has been addressed in some studies, the new vertex interpolation algorithm can be extended to 3D cases naturally. More interesting is that the solvability of the corresponding local system is proved under some assumptions. Additionally, we modify the edge flux approximation by an edge-based discretization of diffusion coefficient, and thus the improved scheme is free of the so-called numerical heat-barrier issue suffered by many existing cell-centered or hybrid schemes. The final scheme allows arbitrary continuous or discontinuous diffusion coefficients and can be applicable to arbitrary star-shaped polygonal meshes. A second-order convergence rate for the approximate solution and a first-order accuracy for the flux are observed in numerical experiments. In the comparative experiments with some existing vertex interpolation algorithms, the new algorithm shows obvious improvement on highly distorted meshes.

AMS subject classifications: 65M08, 65M22

Key words: Cell-centered scheme, nonlinear diffusion equation, vertex interpolation algorithm, linearity-preserving criterion, numerical heat-barrier issue, finite volume scheme.

1. Introduction

Diffusion problems arise in a wide range of applications, such as radiation hydrodynamics (RHD), magnetohydrodynamics (MHD), plasma physics, reservoir modeling,

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and so on. In many applications, certain factors make it difficult to solve diffusion problem, such as the anisotropic and nonlinear diffusion coefficients, the distorted meshes and the multi-materials, etc. Finite volume method is one of the most popular methods to solve diffusion problems because of its simplicity and local conservation. A desirable finite volume scheme is expected to be able to deal with the aforementioned challenges, and to possess as many as possible the good numerical properties, including stability, high order accuracy, local stencil, positivity-preserving property, robustness, efficiency, symmetry and positive definiteness of the resulting linear system, and so on. To our knowledge, there exists no scheme satisfying all the properties mentioned above.

In recent decades, there has been an impressive amount of work on the finite volume schemes of diffusion problems. The readers are referred to [7, 12, 14, 19] and the references cited therein for recent developments. Here we are concerned with a linearity-preserving nine-point scheme (NPS) studied in [35]. NPS was originally proposed in [21] and has been used a long time in RHD applications [10, 11, 25]. NPS has both cell-centered unknowns and vertex unknowns. Generally, the vertex unknowns are treated as auxiliary ones and have to be eliminated to get a pure cell-centered scheme. Interpolation of the vertex unknowns via the cell-centered ones is the key ingredient for NPS.

To our knowledge, there exists three simple interpolation algorithms: arithmetic average interpolation, inverse distance weighted interpolation and linear interpolation [20, 21, 23, 31]. These interpolation algorithms do not satisfy the so-called *linearity-preserving criterion* [35], which requires that each step of the derivation of interpolation algorithm is exact or linearly exact, i.e. exact in the sense whenever the solution is a linear function and the diffusion coefficient is a constant tensor. As a result, all the above three algorithms do not have the second order accuracy and are seldom used nowadays. Least square interpolation was proposed in [9] and a generalization was suggested in [24] to get a more accurate reconstruction of the vertex unknowns on Neumann boundary. Both weights in [9, 24] won't achieve optimal accuracy for problems with discontinuous diffusion coefficients. [16] proposed two types of linearity-preserving explicit interpolation algorithms, the second of which, called LPEW2, has been examined by many authors [8, 18, 29, 32, 33]. However, all aforementioned interpolation algorithms can not be directly extended to 3D cases. As far as we know, all existing vertex interpolation algorithms for 3D diffusion problems are based on the so-called harmonic averaging point [3, 15, 17]. It has been demonstrated in [38] that the harmonic averaging point can go outside the relevant cell face, which results in a possible loss of accuracy for discontinuous and anisotropic diffusion problems. Moreover, since the harmonic averaging point also depends on the diffusion coefficients, its location changes in each nonlinear iteration when nonlinear diffusion problems are solved, leading to a dynamic stencil for the relevant scheme.

Two kinds of vertex interpolation algorithms based on MPFA-O method and extrapolation technology were present in [35,37]. On highly distorted meshes, some schemes employing these two algorithms, such as NPS, have low accuracy and unstable performance, and even produce totally wrong numerical results.