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Efficient and Accurate Legendre Spectral Element Methods for One-Dimensional Higher Order Problems

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Abstract. Efficient and accurate Legendre spectral element methods for solving one-dimensional higher order differential equations with high oscillatory or steep gradient solutions are proposed. Some Sobolev orthogonal/biorthogonal basis functions corresponding to each subinterval are constructed, which reduce the non-zero entries of linear systems and computational cost. Numerical experiments exhibit the effectiveness and accuracy of the suggested approaches.

AMS subject classifications: 35G31, 35Q53, 65M70, 65N35, 33C45

Key words: Legendre spectral element methods, higher order differential equations, Sobolev orthogonal/biorthogonal basis functions, high oscillatory or steep gradient solutions.

1. Introduction

The spectral method possesses high-order accuracy and is becoming an attractive method. However, the global property of the basis functions oftentimes limits its accuracy to problems of high-frequency oscillations or steep gradients. A reasonable way is to use the spectral element method to approximate such problems. The spectral element method, which has the flexibility for arbitrary h and p adaptivity, greatly promotes the development of classical spectral method and has become a popular approach for numerical simulations of fluid dynamics, atmospheric modeling, etc., see, e.g., [3, 5, 7, 13, 14, 16, 20, 21, 23].

The Legendre spectral element method has been widely applied to numerical simulations of partial differential equations. For instance, Rønquist and Patera [15] presented a Legendre spectral element method for solution of multi-dimensional unsteady change-of-phase Stefan problems, Giraldo and Warburton [6] constructed a nodal triangle-based spectral element method for the shallow water equations on the sphere,

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Yu and Guo [24] investigated Legendre spectral element method for fourth order problems with mixed inhomogeneous boundary conditions, Chen *et al.* [4] discussed the Legendre spectral element method for the optimal control problem, Zeng *et al.* [26] developed an alternating direction implicit Legendre spectral element method for the two-dimensional Schrödinger equation, Zhuang *et al.* [27] studied a coupled Legendre-Laguerre spectral element method for the Stokes and Navier-Stokes equations in unbounded domains, Zampieri and Pavarino [25] proposed conforming Legendre spectral element method in space and the second order leap-frog method in time for the acoustic wave equation, Xu and Maday [22] presented Legendre spectral element method for the computation of fluid flows governed by the incompressible Euler equations in a complex geometry, Khan *et al.* [8] introduced Legendre spectral element method with least-square formulation for parabolic interface problems, Mao and Shen [12] constructed Legendre spectral element approximation with geometric mesh for two-sided fractional differential equations.

The aim of this paper is to construct Legendre spectral element methods for higher order differential equations with highly oscillatory or steep gradient solutions. We focus on the design of fast algorithms for second-, third- and fourth- order elliptic equations and some time-dependent problems. As is well known, the classical Legendre spectral element methods for higher order problems usually lead to sparse linear algebraic systems (e.g., for the fourth-order problem (5.1), the coefficient matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ in (5.24) are nine-diagonal, tridiagonal and sparse, respectively), and the total degree of freedom of the resulting algebraic systems increases rapidly. In order to get an effective algorithm, we need to choose appropriate trial and test functions, such that the non-zero entries of the resulting linear system are as few as possible. Recently, Liu *et al.* [10] constructed the Laguerre Sobolev orthogonal basis functions and derived the fully diagonalized algebraic systems for second-order elliptic equations on the half line, Ai *et al.* [1] introduced the Legendre Sobolev orthogonal basis functions and applied them to some elliptic equations on bounded intervals, Li *et al.* [9] proposed the Legendre Sobolev biorthogonal basis functions and developed Legendre dual-Petrov-Galerkin methods for odd-order differential equations. Based on these ideas, we shall in this paper construct some new basis functions corresponding to the subintervals, such that they are Sobolev orthogonal or biorthogonal, and apply them to Legendre spectral element discretizations of the higher order differential equations. Numerical experiments exhibit the efficiency and accuracy of our new algorithms.

The main advantages of the suggested algorithms include:

(i) We can obtain quite accurate numerical results for higher order differential equations with highly oscillatory or steep gradient solutions, by adjusting the mesh step size and/or approximation order;

(ii) The suggested algorithms reduce the non-zero entries of the resulting algebraic systems (e.g., for the fourth-order problem (5.1), the leading coefficient matrix $\mathbb{A}$ in (5.24) is diagonal). Numerical experiments show our new algorithms take less CPU time than the classical spectral element method.