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Superconvergent Cluster Recovery Method for the Crouzeix-Raviart Element

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Abstract. In this paper, we propose and numerically investigate a superconvergent cluster recovery (SCR) method for the Crouzeix-Raviart (CR) element. The proposed recovery method reconstructs a C^0 linear gradient. A linear polynomial approximation is obtained by a least square fitting to the CR element approximation at certain sample points, and then taken derivatives to obtain the recovered gradient. The SCR recovery operator is superconvergent on uniform mesh of four patterns. Numerical examples show that SCR can produce a superconvergent gradient approximation for the CR element, and provide an asymptotically exact error estimator in the adaptive CR finite element method.

AMS subject classifications: 65N15, 65N30

Key words: Crouzeix-Raviart element, gradient recovery, superconvergent cluster recovery.

1. Introduction

In scientific computing, post-processing is an important technique to calculate or recover the information that has physical meanings such as flux, stress, or the other quantities of interest, from the primary numerical approximation. Gradient recovery method is a post-processing technique that reconstructs numerical approximations from finite element solutions to obtain the improved gradient. A classical gradient recovery technique is the simple averaging technique which is as old as the finite element method itself.

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For the C^0 Lagrange element, different kinds of post-processing techniques are developed based on weighted averaging [5, 11, 12, 21], local or global projections [3, 19], post-processing interpolation [24], and the local least-squares methods including the superconvergent patch recovery (SPR) [31], the polynomial preserving recovery (PPR) [25, 29] and the superconvergent cluster recovery (SCR) [22, 26].

The practical usage of the recovery technique is not only to improve the quality of the approximation, but also to construct a posteriori error estimators in adaptive computation. If the recovered gradient superconverges to the exact one, the corresponding recovery type a posteriori error estimator is not only equivalent to the exact error, but also asymptotically exact. Gradient recovery techniques are widely used in engineering practice for its robustness as a posteriori error estimator, its superconvergence of the recovered derivatives, and its efficiency in implementation. The basic idea of the recovery type a posteriori error estimate is to reconstruct a "better" approximation from the original finite element solution by means of the so called recovery technique. As forenamed, if the recovered gradient enjoys the superconvergence property, then it can be used in building an asymptotically exact recovery type a posteriori error estimator. The recovery type a posteriori error estimate was first introduced by Zienkiewicz and Zhu [30, 31], they introduced a posteriori error estimator based on the SPR-recovered gradient. Later, Zhang and Naga developed the PPR based error estimators [25]. Chen *et al.* [13, 14] proposed a SCR based error estimator and applied it in adaptive finite element methods for the Allen–Cahn equation and Cahn–Hilliard equation.

However, all the recovery methods above are limited to C^0 finite element methods. There are relatively few works on the gradient recovery for the nonconforming elements in the literature. The CR element was first proposed in [16] to solve stationary Stokes problem. It was also used to solve elliptic problem [4]. Concerning gradient recovery for the CR element, [10, 11] studied some averaging or projection recovery postprocessing techniques and its application in the posteriori error analysis. [17] researched four flux reconstruction methods for the CR element. Recently, [18] developed a PPR type gradient recovery method for the CR element. The proposed method fits a quadratic polynomial in the sense of least squares at any edge center and then takes derivative to get recovered gradient. Based on an averaging method developed for the Raviart-Thomas element by Brandts [6], [20, 28] obtained the superconvergent results between the exact gradient and the postprocessing gradient for the second order elliptic equation on the uniform triangular mesh and [23] proved superconvergence of CR element by averaging on the non-uniform triangular meshes.

In this paper, we propose and analyze a SCR type gradient recovery method for the CR element. Given a CR finite element approximation, the new method reconstructs a C^0 linear gradient. For any mesh vertex, the recovery method finds some edge centers in a local element patch as the sampling points, which locate around the considering vertex as symmetrical as possible, then fits a linear polynomial to solution values at the sampling points, and takes derivatives to obtain the recovered gradient at the considering point. Similarly to SCR for C^0 finite element methods, the recovery procedure is much simpler and more efficient. We shall demonstrate that the new method is super-