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Low Regularity Error Analysis for Weak Galerkin Finite Element Methods for Second Order Elliptic Problems

Xiu Ye[1,*] and Shangyou Zhang[2]

[1] *Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA*
[2] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*

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Abstract. This paper presents error estimates in both an energy norm and the L^2-norm for the weak Galerkin (WG) finite element methods for elliptic problems with low regularity solutions. The error analysis for the continuous Galerkin finite element remains same regardless of regularity. A totally different analysis is needed for discontinuous finite element methods if the elliptic regularity is lower than H-1.5. Numerical results confirm the theoretical analysis.

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Key words: Weak Galerkin, finite element methods, weak gradient, second-order elliptic problems, low regularity.

1. Introduction

The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. The WG method was first introduced in [16] and then has been applied to solve various partial differential equations such as second order elliptic equations, biharmonic equations, Stokes equations, convection dominant problems, two-phase flow problems and Maxwell's equations [1,2,4–8,10–14,17–19]. However, the standard a priori error analysis of weak Galerkin finite element methods requires additional regularity on solutions. For second order elliptic problems, it is usually assumed that the solutions are in H^{1+s} with $s > \frac{1}{2}$. It is desirable to develop a new type of error estimates for the problems with low regularity solutions such as elliptic interface problems.

*Corresponding author. *Email addresses:* xxye@ualr.edu (X. Ye), szhang@udel.edu (S. Zhang)

In [3], a new error estimate in energy norm with low regularity assumption has been developed for the interior penalty discontinuous Galerkin methods. The purpose of this work is to provide an error estimate in an energy norm for elliptic problems with low regularity solutions for the WG methods following the ideas in [3]. In addition, we also derive a convergence analysis in the L^2 norm with low regularity assumption for the weak Galerkin finite element method.

We consider the following elliptic problem that seeks an unknown function u satisfying

$$-\nabla\cdot(a\nabla u)=f \quad \text{in } \Omega, \tag{1.1}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polytopal domain in $\mathbb{R}^d$.

2. Weak Galerkin finite element schemes

We adopt standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$ for $s\geq 0$. When $D=\Omega$, we drop the subscript D in the norm and inner product notation.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of triangles and tetrahedrons. Denote by $\mathcal{E}_h$ the set of all edges and faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0=\mathcal{E}_h\backslash\partial\Omega$ be the set of all interior edges and faces. For every element $T\in\mathcal{T}_h$, we denote by h_T its diameter and mesh size $h=\max_{T\in\mathcal{T}_h}h_T$ for $\mathcal{T}_h$. We adopt the following notations,

$$(v,w)_{\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}(v,w)_T=\sum_{T\in\mathcal{T}_h}\int_T vwdx,$$

$$\langle v,w\rangle_{\partial\mathcal{T}_h}=\sum_{T\in\mathcal{T}_h}\langle v,w\rangle_{\partial T}=\sum_{T\in\mathcal{T}_h}\int_{\partial T} vwds.$$

For a given integer $k\geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

$$V_h=\big\{v=\{v_0,v_b\}: v_0|_T\in P_k(T),\ v_b|_e\in P_k(e),\ e\subset\partial T, T\in\mathcal{T}_h\big\} \tag{2.1}$$

and its subspace V_h^0 is defined as

$$V_h^0=\big\{v: v\in V_h,\ v_b=0 \text{ on } \partial\Omega\big\}. \tag{2.2}$$

We would like to emphasize that any function $v\in V_h$ has a single value v_b on each edge $e\in\mathcal{E}_h$.

For $v=\{v_0,v_b\}\in V_h$ or $v\in H_0^1(\Omega)$, a weak gradient $\nabla_w v$ is a piecewise vector valued polynomial such that on each $T\in\mathcal{T}_h$, $\nabla_w v\in[P_{k-1}(T)]^d$ satisfies

$$(\nabla_w v,\mathbf{q})_T=-(v_0,\nabla\cdot\mathbf{q})_T+\langle v_b,\mathbf{q}\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\mathbf{q}\in[P_{k-1}(T)]^d. \tag{2.3}$$