

# Generalized Singular Value Decompositions for Tensors and Their Applications

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**Abstract.** In this paper, we consider the generalized singular value decompositions for two tensors via the T-product. We investigate and discuss in detail the structures of the quotient singular value decomposition (T-QSVD) and product singular value decomposition (T-PSVD) for two tensors. The algorithms are presented with numerical examples illustrating the results. For applications, we consider color image watermarking processing with T-QSVD and T-PSVD. There are two advantages to T-QSVD and T-PSVD approaches on color watermark processing: two color watermarks can be processed simultaneously and only one key needs to be saved.

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## 1. Introduction

Tensor decompositions arise in a wide variety of application areas, for example, in genomic signals, data mining, signal processing, eigenvalues and so on [3, 7, 8, 32, 34, 42, 43]. Various tensor decompositions under different tensor products have been investigated to develop the extension of linear algebra methods to this multilinear context (e.g. [2, 9, 21, 28, 40]). Tensor decomposition under t-product is a new decomposition of tensors. Since Kilmer and Martin [27] first considered tensor SVD (T-SVD) under t-product for third-order tensors in 2011, there have been many papers discussing T-SVD (e.g. [17, 25, 31, 46]). The necessary theoretical framework for T-SVD computation was set up in [26]. The property, algorithm and applications of T-SVD for  $p$ -order tensor

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were derived in [29]. The relationship between T-SVD and other decompositions was discussed in [44].

Generalized singular value decomposition (GSVD) is a fundamental theme in linear algebra and has applications in signal processing [12], genomic signal processing [35, 36], generalized eigenvalue problems [47], discrete linear ill-posed problems [16], information retrieval [24], etc. The quotient singular value decomposition (QSVD) and product singular value decomposition (PSVD) are two important kinds of generalized singular value decompositions. QSVD was first considered in [41] and then refined in [38]. The standardized nomenclature of this decomposition is given in [10]. A variational formulation for QSVD is presented in [4]. PSVD was first proposed in [22], and further discussed in [15]. The nonuniqueness of the factorization factors in the PSVD is characterized in [5]. For more information on the QSVD and PSVD see [1, 6, 11, 13, 14, 19, 20, 23, 37, 45].

Note that most of the research work related to T-SVD builds upon the existing work for one tensor case. The literature on generalized singular value decomposition for two tensors under t-product is limited. Motivated by the wide application of tensor decompositions and t-product and in order to improve the theoretical development of GSVD for tensors, we consider the quotient singular value decomposition (T-QSVD) and product singular value decomposition (T-PSVD) for two tensors under t-product.

The main contribution of the paper is twofold. Firstly, we investigate and discuss in detail the structures and algorithms of T-QSVD and T-PSVD. Secondly, we illustrate an application in color image watermarking processing where the T-QSVD and T-PSVD are used.

The remainder of the paper is organized as follows. In Section 2, we review the definitions of tensors, t-product, identity tensor, conjugate transpose, inverse tensor and unitary tensor. In Sections 3 and 4, we investigate and analyze in detail the structure and properties of T-QSVD and T-PSVD. We give algorithms and numerical examples to illustrate the main results. In Section 5, we derive an application on color image watermarking processing.

## 2. Preliminaries

A third-order tensor  $\mathcal{A} = (a_{i_1 i_2 i_3})_{n_1 \leq i_j \leq n_j}$  ( $j = 1, 2, 3$ ) is a multidimensional array with  $n_1 n_2 n_3$  entries. Let  $\mathbb{C}^{n_1 \times n_2 \times n_3}$  stands for the set of the order three dimension  $n_1 \times n_2 \times n_3$  tensors over the complex number field  $\mathbb{C}$ .

If  $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathbb{C}^{n_1 \times n_2 \times n_3}$  with  $n_1 \times n_2$  frontal slices  $A_1, \dots, A_{n_3}$ , then

$$\text{unfold}(\mathcal{A}) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n_3} \end{bmatrix}, \quad (2.1)$$