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An Acceleration Technique for the Augmented IIM for 3D Elliptic Interface Problems

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Abstract. A new fast algorithm based on the augmented immersed interface method and a fast Poisson solver is proposed to solve three dimensional elliptic interface problems with a piecewise constant but discontinuous coefficient. In the new approach, an augmented variable along the interface, often the jump in the normal derivative along the interface is introduced so that a fast Poisson solver can be utilized. Thus, the solution of the Poisson equation depends on the augmented variable which should be chosen such that the original flux jump condition is satisfied. The discretization of the flux jump condition is done by a weighted least squares interpolation using the solution at the grid points, the jump conditions, and the governing PDEs in a neighborhood of control points on the interface. The interpolation scheme is the key to the success of the augmented IIM particularly. In this paper, the key new idea is to select interpolation points along the normal direction in line with the flux jump condition. Numerical experiments show that the method maintains second order accuracy of the solution and can reduce the CPU time by 20-50%. The number of the GMRES iterations is independent of the mesh size.

AMS subject classifications: 65N06, 65N50

Key words: 3D elliptic interface problem, augmented IIM, fast Poisson solver, directional least squares interpolation.

1. Introduction

The augmented immersed interface method (AIIM) was first presented in [11] for solving elliptic interface problems with a piecewise constant coefficient. Since then, the

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augmented approach has been studied and extended for various applications in two dimensions [10, 12] and three dimensions [5, 6]. The method in [11] was designed by introducing an augmented variable in the jump of the normal derivative to take advantage of existing fast Poisson solvers. In discretization, the approximate solution of the original problem and the augmented variable are the solutions of a slightly larger linear system than the one for a regular Poisson equation. With a block elimination, the Schur complement system for the augmented variable can be derived. Then the GMRES iteration can be utilized to update the augmented variable. It has been proved that the number of iterations is nearly independent of the mesh size. In practice, the Schur complement matrix does not need to be evaluate explicitly. One matrix-vector multiplication is equivalent to solve a Poisson equation plus an interpolation scheme to evaluate the residual of flux jump conditions. Thus, the method is efficient and fast in general, which has been shown for two dimensional problems.

A closely relevant problem, solving elliptic PDEs on irregular domains can also be solved using the AIIM by an embedding technique. An irregular domain can be embedded in a larger regular domain and the original boundary value problem can be reformulated as an interface problem with the boundary condition as an augmented equation. Elliptic interface problems with piecewise constant coefficients and problems on irregular domains are often encountered in fluid dynamics, material science, such as the heat conduction in different materials with different conductivities, incompressible Navier-Stokes equations on irregular domains [13], concentration equations with discontinuous diffusion coefficients for chemical vapor infiltration process and so forth.

Elliptic interface problems can be solved by various methods such as body fitted finite element methods (the construction of a body-fitting mesh is nontrivial and maybe time-consuming), the immersed boundary method (IBM), the immersed finite element method (IFEM), the extended finite element method (XFEM), the ghost fluid method (GFM), the matched interface and boundary method (MIB), the Augmented MIB, and some others, see [2, 4, 7, 9, 10, 14–20] for an incomplete list.

There are two main components in implementation of an AIIM: one is solving the Poisson equation with known jump in the solution and the jump in the normal derivative that is the augmented variable; the other is finding the residual of the constraint (which is often the jump condition) using the computed approximate solution to get a better approximation to the augmented variable. The accuracy of the interpolation scheme for computing the augmented variable in the jump of the normal derivative turns out to be a crucial step in the implementation. This is the key to the success of our new accelerated AIIM. The least squares interpolation (under-determined local small linear systems by using grid points in the neighborhood of the interface or boundary) and the singular value decomposition (SVD) solver are utilized to determine the coefficients of the interpolation scheme.

We consider the following 3D elliptic interface problem with a piecewise constant coefficient

$$\nabla \cdot \big(\beta(x,y,z)\nabla u\big) = f(x,y,z) \quad \text{in} \quad \Omega\backslash\Gamma, \tag{1.1}$$