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Accelerated Non-Overlapping Domain Decomposition Method for Total Variation Minimization

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Abstract. We concern with fast domain decomposition methods for solving the total variation minimization problems in image processing. By decomposing the image domain into non-overlapping subdomains and interfaces, we consider the primal-dual problem on the interfaces such that the subdomain problems become independent problems and can be solved in parallel. Suppose both the interfaces and subdomain problems are uniformly convex, we can apply the acceleration method to achieve an $\mathcal{O}(1/n^2)$ convergent domain decomposition algorithm. The convergence analysis is provided as well. Numerical results on image denoising, inpainting, deblurring, and segmentation are provided and comparison results with existing methods are discussed, which not only demonstrate the advantages of our method but also support the theoretical convergence rate.

AMS subject classifications: 68U10, 65M55, 65N55

Key words: Non-overlapping domain decomposition method, primal-dual algorithm, total variation, Rudin-Osher-Fatemi model, Chan-Vese model.

1. Introduction

Minimizing the total variation (TV) was first proposed by Rudin *et al.* [29] for image denoising problem, and has captured wide attention due to its ability in preserving sharp edges and discontinuities when removes noises. Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundaries, $f:\Omega\rightarrow\mathbb{R}$ be a given image defined on the domain Ω, and $u:\Omega\rightarrow\mathbb{R}$ be the latent clean image. The generalized TV minimization model can be formulated to minimize the following energy functional:

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$$\min_{u \in \mathrm{BV}(\Omega)} \left\{ F(u) := \mathrm{TV}(u) + \frac{\lambda}{2} \int_\Omega (Au - f)^2 dx \right\}, \tag{1.1}$$

where $\lambda \equiv$ const > 0 is a weight parameter used to tradeoff between the data fidelity term and the regularization term, A is a linear and bounded operator varying with image processing tasks, $\mathrm{BV}(\Omega)$ is the space of functions of bounded variation on Ω, and $\mathrm{TV}(u)$ stands for the total variation defined by

$$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \boldsymbol{p} dx : \boldsymbol{p} = (p_1, p_2) \in C_0^1\big(\Omega; \mathbb{R}^2\big),\ \|\boldsymbol{p}\|_\infty \leq 1 \right\}$$

with $C_0^1(\Omega, \mathbb{R}^2)$ being the space of continuously differentiable vector valued functions with compact support on Ω and

$$\|\boldsymbol{p}\|_\infty = \sup_x \sqrt{\sum_i p_i^2(x)}.$$

Various numerical algorithms have been studied for solving the TV minimization problem, especially for the Rudin-Osher-Fatemi model, including the direct primal approaches such as the gradient descent method [29], fixed-point method [33], split Bregman iteration [18], and augmented Lagrangian method [34]. Chambolle [3] reformulated (1.1) by the Fenchel-Rockafellar dual and solved the dual problem by the semi-implicit gradient descent algorithm. Chambolle and Pock [4] considered the min-max optimization problem for solving the general problems in image processing, where the first-order primal-dual algorithm was developed for the nonlinear convex problem with an $\mathcal{O}(1/n)$ convergent rate of convergence. What is more, as long as the minimization problem is uniformly convex, e.g., the Rudin-Osher-Fatemi model, it is shown that the $\mathcal{O}(1/n^2)$ convergence rate can be achieved by updating the step sizes dynamically. Other methods for solving the model (1.1) include the fast non-iterative algorithm in [11], the primal-dual fixed-point algorithm in [9], the proximity algorithm in [27], and general Douglas-Rachford algorithms in [12], etc.

The aforementioned methods work well on small- and medium-scale image problems, but fail to address extremely large problems in realistic CPU-time such as traffic problems [32]. Domain decomposition methods (DDMs) [31,35] can make use of distributed memory computers by breaking down the problem into a sequence of smaller scale subproblems and solve them in parallel. Over the past two decades, both overlapping and non-overlapping DDMs have been well studied for the variational model in image processing problems. Fornasier and Schönlieb [17] proposed a non-overlapping DDM algorithm for total variation minimization, the convergence of which was theoretically guaranteed. The idea was further studied for the case of overlapping DDM in [16]. Xu *et al.* [36] proposed a two-level overlapping DDM for the Rudin-Osher-Fatemi model by directly solving the nonlinear partial differential equation. Duan and Tai [15] developed an overlapping DDM for the Rudin-Osher-Fatemi model, where graph cuts were used to solve the subdomain minimization problem. To avoid the difficulties in minimizing the nonsmooth and nonadditive total variation term, the dual