Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2021-0040

Vol. **14**, No. 4, pp. 1068-1084
November 2021

On a Parabolic Sine-Gordon Model

Xinyu Cheng[1], Dong Li[3], Chaoyu Quan[2,3] and Wen Yang[4,5,*]

[1] *Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada*
[2] *SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China*
[3] *Department of Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China*
[4] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, P.R. China*
[5] *Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, Wuhan 430071, P.R. China*

Received 20 March 2021; Accepted (in revised version) 20 May 2021

Abstract. We consider a parabolic sine-Gordon model with periodic boundary conditions. We prove a fundamental maximum principle which gives a priori uniform control of the solution. In the one-dimensional case we classify all bounded steady states and exhibit some explicit solutions. For the numerical discretization we employ first order IMEX, and second order BDF2 discretization without any additional stabilization term. We rigorously prove the energy stability of the numerical schemes under nearly sharp and quite mild time step constraints. We demonstrate the striking similarity of the parabolic sine-Gordon model with the standard Allen-Cahn equations with double well potentials.

AMS subject classifications: 35K55, 65M12, 65M22

Key words: Sine-Gordon equation, backward differentiation formula, implicit-explicit scheme.

1. Introduction

In this work we are concerned with the following parabolic sine-Gordon equation:

$$\begin{cases} \partial_t u = \kappa^2 \Delta u + \sin u, & (t,x) \in (0,\infty) \times \Omega, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* xycheng@math.ubc.ca (X. Cheng), quancy@sustech.edu.cn (C. Quan), mpdongli@gmail.com (D. Li), wyang@wipm.ac.cn (W. Yang)

 ©2021 Global-Science Press

where κ^2 is the diffusion constant and Ω is either a periodic torus $\mathbb{T} = [-\pi, \pi]$ in 1D or the torus $\mathbb{T}^2 = [-\pi, \pi] \times [-\pi, \pi]$ in 2D. The unknown function $u : \Omega \to \mathbb{R}$ typically represents the concentration difference in the phase field context. For smooth solutions, the basic energy associated with (1.1) is

$$E(u) = \int_\Omega \left(\frac{\kappa^2}{2} |\nabla u|^2 + \cos u \right) dx. \tag{1.2}$$

The fundamental energy conservation law takes the form

$$\frac{d}{dt} E(u) + \int_\Omega |\partial_t u|^2 dx = 0. \tag{1.3}$$

It follows that

$$E\big(u(t)\big) \le E\big(u(s)\big) \quad \forall\, t \ge s, \tag{1.4}$$

which gives a priori control of the homogeneous $\dot{H}^1$-norm of the solution. Better estimates are also available. For example assuming u_0 is bounded, then by using the fact that the nonlinear term $\sin u$ is bounded by 1, one can show that the solution remains bounded for all finite time. Bootstrapping from this then easily yields global wellposedness and regularity of the solution. Somewhat akin to the Eq. (1.1) is the following slightly more general model:

$$\partial_\tau v = \kappa^2 \Delta v + \gamma \sin \beta v, \tag{1.5}$$

where $\beta > 0$, $\gamma > 0$ are parameters, and we denote by τ the time variable. One can rewrite (1.5) as

$$\frac{\partial(\beta v)}{\partial(\gamma \beta \tau)} = \frac{\kappa^2}{\gamma \beta} \Delta(\beta v) + \sin(\beta v). \tag{1.6}$$

Consequently a change of variable $u = \beta v$, $t = \gamma \beta \tau$ transforms (1.5) into the standard form (1.1).

The classical one dimensional sine-Gordon equation

$$\partial_{tt}\phi - \partial_{xx}\phi = -\sin \phi \tag{1.7}$$

dates back at least to Frenkel and Kontorova [8] who considered the motion of a slip in an infinite chain of atoms lying on top of a given fixed chain of alike atoms. To study the propagation of the slip they obtained a difference differential equations which was approximated by the sine-Gordon equation (1.7). In the realm of nonlinear field theory, the sine-Gordon equation

$$\partial_{tt}\phi - \partial_{xx}\phi = -m^2 \sin \phi \tag{1.8}$$

arises as one of the simplest intrinsically nonlinear theories. The classical point-like particle theories suffer divergence problems such as the well-known self-energy problem of electrodynamics. It was realized that (cf. the discussion on [1, pp. 260]) one