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Strong Convergence of a Fully Discrete Scheme for Multiplicative Noise Driving SPDEs with Non-Globally Lipschitz Continuous Coefficients

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Abstract. This work investigates strong convergence of numerical schemes for nonlinear multiplicative noise driving stochastic partial differential equations under some weaker conditions imposed on the coefficients avoiding the commonly used global Lipschitz assumption in the literature. Space-time fully discrete scheme is proposed, which is performed by the finite element method in space and the implicit Euler method in time. Based on some technical lemmas including regularity properties for the exact solution of the considered problem, strong convergence analysis with sharp convergence rates for the proposed fully discrete scheme is rigorously established.

AMS subject classifications: 60H15, 60H35, 65C30, 65M60

Key words: Stochastic partial differential equations, strong convergence, non-global Lipschitz, finite element method, variational solution, mean square error estimate.

1. Introduction

This work is devoted to the numerical approximation of the following initial boundary value problem of the Itô-type stochastic partial differential equations (SPDEs)

$$\begin{cases} \mathrm{d}u(t,x) - \Delta u(t,x)\,\mathrm{d}t = \widetilde{f}\big(x,u(t,x)\big)\mathrm{d}t \\ \quad +\widetilde{g}\big(x,u(t,x)\big)\mathrm{d}W(t), \quad x \in D, \quad t\in(0,T], \\ u(t,x)=0, \qquad\qquad\qquad \text{on } \ \partial D\times[0,T], \\ u(0,x)=u_0(x), \qquad\qquad \text{in } \ D\times\{0\}, \end{cases} \tag{1.1}$$

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where T is a fixed positive constant, D is a bounded convex domain in $\mathbb{R}^d, d = 1, 2, 3,$ with polygonal boundary ∂D, Δ is the Laplacian operator, $\widetilde{f}, \widetilde{g}\colon D \times \mathbb{R} \to \mathbb{R}$ are two appropriate regular functions, and the driving noise W is characterized by a standard $\mathbb{R}$-valued Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with normal filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$. The above problem (1.1) should be interpreted mathematically as a stochastic integral equation and particularly when u is a strong solution of (1.1) we have

$$u(t,x) - \int_0^t \Delta u(s,x)\,\mathrm{d}s = u_0(x) + \int_0^t \widetilde{f}\big(x, u(s,x)\big)\mathrm{d}s + \int_0^t \widetilde{g}\big(x, u(s,x)\big)\mathrm{d}W(s)$$

for $t \in [0, T]$ and $x \in D$, where the last integral with respect to the Wiener process is an Itô stochastic integral. Precise descriptions on the coefficient functions $\widetilde{f}$ and $\widetilde{g}$, and the initial-value function u_0 will be given in Section 2.

SPDEs have been found numerous applications in various branches of applied sciences, ranging from chemistry, physics and biology to engineering and economics. In recent years the theory of stochastic partial differential equations has already had an intensive development, see [6,8,22,30,31] and references therein. Generally, there are basically two approaches to analyzing SPDEs: the semigroup approach and variational approach. As far as the first approach is concerned, we refer to the monograph [8], where the semigroup approach is extensively studied and the corresponding solution is called mild solution. For the second approach, the equation is usually considered in a Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ of Hilbert spaces with the space V as the domain of the unbounded operator and V^* its dual. In this case variational solutions are produced [6, 21, 22, 30, 31]. The variational solutions of SPDEs are studied mainly under the so-called coercivity and monotonicity conditions, which are commonly assumed in the study of deterministic partial differential equations (PDEs). In this paper, we will consider the problem in the framework of the variational approach. There are three main reasons for this choice. One is that under some stronger assumptions, such as the conditions of the input data u_0, coercivity and monotone nonlinearity, the variational solution can have higher order regularity [6], while the regularity of solutions will be decreasing when using the semigroup method to deal with SPDEs [41]. The second reason is built on the fact that the semigroup approach has a very restrictive requirement on the unbounded operator, which may not be applicable for some general cases, such as the case when the unbounded operator is time-dependent. Hence by using variational approach we are capable of handling some more general SPDEs. The third reason is that in the framework of the variational approach, one can make good use of the Itô formula, which plays a powerful role in our following analysis.

Since only a few and very simple SPDEs can be solved analytically, it is of great significance to develop accurate and efficient numerical methods to solve SPDEs. And the study of numerical approximation of SPDEs is a very active ongoing research area and has attracted considerable attention. Particularly, in the past two decades, plenty of works have been done on the study of numerical methods for SPDEs. Based on the mild solution approach, one is referred to [12, 20, 27, 28, 37–39] and references