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Order Reduced Schemes for the Fourth Order Eigenvalue Problems on Multi-Connected Planar Domains

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Abstract. In this paper, we study the order reduced finite element method for the fourth order eigenvalue problems on multi-connected planar domains. Particularly, we take the biharmonic and the Helmholtz transmission eigenvalue problems as model problems, present for each an equivalent order reduced formulation and a corresponding stable discretization scheme, and present rigorous theoretical analysis. The schemes are readily fit for multilevel correction algorithms with optimal computational costs. Numerical experiments are given for verifications.

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1. Introduction

The fourth order eigenvalue problems are widely used in applied sciences with various sources. For example, the biharmonic eigenvalue problem is the most fundamental fourth order model problem and can be used directly for modeling the vibration of thin plates. The Helmholtz transmission eigenvalue problem is an indirect example, which has important applications in a variety of inverse problems, such as, target iden-

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tification and nondestructive testing [17]. The latter example is originally a system of second order problems and can be rewritten to be a fourth order nonlinear eigenvalue problem to avoid non-physical eigenspaces [18]. In this paper, we focus ourselves on these two model problems.

For biharmonic eigenvalue problem, there has been a long history on developing finite element methods and many schemes have been proposed for discretization [2, 6, 7, 13, 21, 26, 39], computation of guaranteed upper and lower bounds [4, 15, 16, 24, 40], and adaptive method and its convergence analysis [11]. For transmission eigenvalue problem, significant effort has been focused on developing effective numerical methods since 2010. The first numerical study may be found in [9] where three finite element methods were proposed. In [27], two iteration methods with rigorous convergence analysis were proposed. Here the author constructs a nonlinear function whose values are generalized eigenvalues of a series of self-adjoint fourth order problems. The roots of the function are the transmission eigenvalues. However, only real eigenvalues can be captured. For its fourth-order problem formulation, many schemes have been constructed, such as the Argyris element method [9], the (multi-level) BFS element method [19], the Morley element method [20, 30], the cubic H^2 nonconforming finite element [32], the modified Zienkiewciz element and the Morley-Zienkiewicz element [38] and other low complexity finite element methods including an interior penalty discontinuous Galerkin method using C^0 Lagrange elements (C^0IPG method) [12], and so on. Besides these discretizations in primal formulations, there are also some order reduced methods for these problems. The related works for mixed element method can be referred to [3, 9, 18, 31, 37]. We note that the famous Ciarlet-Raviart mixed scheme [8] is frequently used. However, sometimes the Ciarlet-Raviart mixed formulation is not equivalent to the original variational problem on H^2 spaces, and we refer to [43] for some numerical observation of spurious modes. Some non-traditional methods, such as the linear sampling method [28] and the inside-out duality [22], were proposed to searching eigenvalues using scattering data. However, the computation tends to be very expensive. There also existed other methods in the literature, such as the recursive integral method (short for RIM) [17, 29] based on the eigenprojectors of compact operators which is designed to approximate all eigenvalues within a specific region without solving eigenvectors.

Accompanied by being a discretization scheme, a very important motivation for establishing order reduced scheme is to implement the optimal multilevel algorithms. The multi-level correction method based on nested essence has become an effective tool in scientific computing. For eigenvalue problems, many multi-level algorithms have been designed and implemented. A type of multi-level scheme is presented by Lin-Xie [25, 33]. The method is related to [23, 34–36], and has presented a framework of designing multi-level schemes which works well for the elliptic eigenvalue problem and stable saddle point problem, provided a series of subproblems with intrinsic nestedness constructed. For the fourth order problem in primal formulations where the second order Sobolev spaces are involved, the discretizations can hardly be nested. The only known nested finite element other than spline type ones is the BFS element