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Stability and Convergence Analyses of the FDM Based on Some L-Type Formulae for Solving the Subdiffusion Equation

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Abstract. Some well-known L-type formulae, i.e., L1, L1-2, and L1-2-3 formulae, are usually employed to approximate the Caputo fractional derivative of order $\alpha \in (0,1)$. In this paper, we aim to elaborate on the stability and convergence analyses of some finite difference methods (FDMs) for solving the subdiffusion equation, i.e., a diffusion equation which exploits the Caputo time-fractional derivative of order α. In fact, the FDMs considered here are based on the usual central difference scheme for the spatial derivative, and the Caputo derivative is approximated by using methods such as the L1, L1-2, and L1-2-3 formulae. Thanks to a specific type of the discrete version of the Gronwall inequality, we show that the FDMs are unconditionally stable in the maximum norm and also discrete H^1 norm. Then, we prove that the finite difference method which uses the L1, L1-2, and L1-2-3 formulae has the global order of convergence $2-\alpha$, $3-\alpha$, and 3, respectively. Finally, some numerical tests confirm the theoretical results. A brief conclusion finishes the paper.

AMS subject classifications: 65M12, 65M06, 26A33

Key words: Stability analysis, order of convergence, Caputo derivative, L1 formula, L1-2 formula, L1-2-3 formula, subdiffusion equation, Gronwall inequality.

1. Introduction

Subdiffusion equations, i.e. fractional diffusion equations with the Caputo time-fractional derivative of order $\alpha \in (0,1)$ have attracted much attention in recent years to model engineering and physical problems [24,31]. Various numerical methods have been constructed to find approximate solutions to such problems that can not be solved

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analytically. Some of them are finite difference method, finite element method, and local discontinuous Galerkin method, see, e.g., [8,10,38,39]. In recent years, developing stable numerical methods has been becoming more and more interesting. An essential part of such numerical schemes is to approximate the Caputo fractional derivative appropriately.

Although there are some formulae based on multistep methods and convolution quadrature [19, 20] such as the backward differentiation formulas (BDFs), we restrict our attention to formulae which are based on interpolation. In this regard following the idea of piecewise linear interpolation, the L1 formula was presented with a numerical accuracy of order $2 - \alpha$ [13, 26]. Murio in [25] constructed a first-order L1 formula using the forward Euler method. To improve the numerical accuracy of classical L1 formula, some high order formulae have been proposed recently. They have used a quadratic interpolation for the integrand function in the Caputo integral, for example, the L1-2, and L2-1_σ methods presented with the numerical accuracy of $3 - \alpha$ by Gao *et al.* [5] and Alikhanov [1], respectively. Lv and Xu [21] constructed a $3 - \alpha$ order formula, which is slightly different from L1-2 and L2-1 methods, with the theoretical result of stability and convergence order. Wang *et al.* [32] applied the idea in [21] and developed a new approximation at the first time-level to obtain an accuracy of global order $3 - \alpha$ with less complicated computations. In [2], based on the quadratic interpolation on each subinterval, a high-order numerical scheme for fractional ordinary differential equations with the Caputo derivative has been introduced. That scheme is unconditionally stable, and for general nonlinear equations, the uniform sharp numerical order $3 - \alpha$ has been rigorously proven for sufficiently smooth solutions at all time steps. In [14], two types of finite difference methods based on linear and quadratic approximations for the numerical solutions of time-fractional parabolic equations with the Caputo time derivative has been presented. Following the idea of piecewise interpolation, Mokhtari *et al.* [22] constructed the L1-2-3 formula based on cubic interpolation. Recently, against L-type formulae, a new class of schemes based on spline interpolations, called S1, S2, and S3 formulae, was proposed to approximate the Caputo fractional derivative with the advantages of global accuracy of orders $2-\alpha$, $3 - \alpha$, and $4 - \alpha$, respectively [27]. The stability and convergence analyses of these schemes are also in progress.

One of the interesting applications of the mentioned formulae is to construct some numerical methods for solving subdiffusion or even superdiffusion equations. Sun and Wu [30] used the L1 schema mixed with a finite difference scheme to solve a diffusion-wave system and proved the unconditionally stability and convergence of this method. Liao *et al.* [16] studied the stability and convergence of a numerical method based on the L1 formula, on the nonuniform mesh grid, for solving linear reaction-subdiffusion equations. To extend the L1 method for initial nonsmooth data, a modified L1 formula for solving time-fractional partial differential equations was introduced by Yan *et al.* [37], and it was shown that the proposed schema has a convergence order of $2-\alpha$ for smooth and nonsmooth initial data in both homogeneous and inhomogeneous cases. Jin *et al.* [7] established a first-order convergence scheme with the L1 formula