

Artificial Boundary Conditions for Time-Fractional Telegraph Equation

Wang Kong^{1,2} and Zhongyi Huang^{3,*}

¹ Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China

² Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 211106, China

³ Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

Received 28 April 2021; Accepted (in revised version) 8 December 2021

Abstract. In this paper, we study the numerical solution of the time-fractional telegraph equation on the unbounded domain. We first introduce the artificial boundaries Γ_{\pm} to get a finite computational domain. On the artificial boundaries Γ_{\pm} , we use the Laplace transform to construct the exact artificial boundary conditions (ABCs) to reduce the original problem to an initial-boundary value problem on a bounded domain. In addition, we propose a finite difference scheme based on the $\mathcal{L}1_{-2}$ formulæ for the Caputo fractional derivative in time direction and the central difference scheme for the spatial directional derivative to solve the reduced problem. In order to reduce the effect of unsmoothness of the solution at the initial moment, we use a fine mesh and low-order interpolation to discretize the solution near $t = 0$. Finally, some numerical results show the efficiency and reliability of the ABCs and validate our theoretical results.

AMS subject classifications: 65M10, 78A48

Key words: Artificial boundary conditions, time-fractional telegraph equation, finite difference scheme, fractional Cattaneo heat conduction law.

1. Introduction

In this paper, we consider the following time-fractional telegraph equation on the unbounded domain $\Omega = \mathbb{R} \times [0, T]$:

$${}_0^C \mathcal{D}_t^{1+\alpha} u(x, t) + {}_0^C \mathcal{D}_t^{\beta} u(x, t) = u_{xx}(x, t) + f(x, t), \quad (x, t) \in \Omega, \quad (1.1a)$$

*Corresponding author. *Email addresses:* zhongyih@mail.tsinghua.edu.cn (Z. Huang), wkong@nuaa.edu.cn (W. Kong)

$$u(x, 0) = \phi(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

$$u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}, \tag{1.1c}$$

$$|u(x, t)| \rightarrow 0, \quad |x| \rightarrow \infty \tag{1.1d}$$

with the parameters $0 < \alpha, \beta < 1$. Here, the initial values $\phi(x), \psi(x) \in C_0^\infty(\mathbb{R})$ and the source term $f(x, t) \in C_{-1}^1([0, T], C_0^\infty(\mathbb{R}))$, that is, there is a real number $p > -1$ such that

$$f(x, t) = t^p f_1(x, t),$$

where $f_1(x, t) \in C^1([0, T], C_0^\infty(\mathbb{R}))$. Besides, ${}_0^C \mathcal{D}_t^{n+\xi}$ denotes the Caputo fractional time derivative of order $n + \xi$,

$${}_0^C \mathcal{D}_t^{n+\xi} u(t) = \frac{t^{-\xi}}{\Gamma(1-\xi)} * \frac{d^{n+1}}{dt^{n+1}} u(t),$$

where ‘*’ denotes the convolution with respect to t

$$f(t) * g(t) = \int_0^t f(s)g(t-s)ds.$$

The time-fractional telegraph equations (1.1) can simulate the heat conduction in rigid conductors [27] or in porcine muscle and blood [18]. The heat conduction process can be described by the energy balance equation

$$\rho c \frac{\partial}{\partial t} T(x, t) = - \frac{\partial}{\partial x} q(x, t)$$

with the heat flux $q(x, t)$ which meets the following fractional Cattaneo heat conduction law:

$$\tau_0 {}_0^C \mathcal{D}^{1-\beta} ({}_0^C \mathcal{D}_t^\alpha q(x, t)) + q(x, t) = -\lambda_0 {}_0^C \mathcal{D}^{1-\beta} \left(\frac{\partial}{\partial x} T(x, t) \right), \tag{1.2}$$

where τ_0 and λ_0 respectively denote the generalized relation time (measured in $s^{1+\alpha-\beta}$) and the generalized thermal conductivity (measured in $\frac{J}{Kms^\beta}$). Actually, $\beta = 1$ leads to the fractional Cattaneo constitutive equation

$$\tau_0 {}_0^C \mathcal{D}_t^\alpha q(x, t) + q(x, t) = -\lambda_0 \frac{\partial}{\partial x} T(x, t). \tag{1.3}$$

If $\alpha = 0$, we can get the Fourier heat conduction law, which describes the heat conduction process under normal conditions; and while $\alpha = 1$, it is Cattaneo’s heat conduction law, which can describe the heat conduction process in the rapidly changing region. The fractional Cattaneo constitutive equation (1.3) is a generalization of the Cattaneo’s heat conduction law, which takes into account the effect of the change history of heat flux, while Cattaneo’s heat conduction law (1.2) adds the effect of the temperature gradient’s change history, thus can describe more complex heat transfer processes.