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The Discontinuous Galerkin Method by Divergence-Free Patch Reconstruction for Stokes Eigenvalue Problems

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Abstract. The discontinuous Galerkin method by divergence-free patch reconstruction is proposed for Stokes eigenvalue problems. It utilizes the mixed finite element framework. The patch reconstruction technique constructs two categories of approximation spaces. Namely, the local divergence-free space is employed to discretize the velocity space, and the pressure space is approximated by standard reconstruction space simultaneously. Benefit from the divergence-free constraint; the identical element patch serves two approximation spaces while using the element pair $\mathbb{P}^{m+1}/\mathbb{P}^m$. The optimal error estimate is derived under the inf-sup condition framework. Numerical examples are carried out to validate the inf-sup test and the theoretical results.

AMS subject classifications: 49N45, 65N21

Key words: Stokes eigenvalue problems, divergence-free, patch reconstruction, discontinuous Galerkin, mixed finite element.

1. Introduction

Due to the various applications in fluid dynamics and structural analysis, the eigenvalue problem of the Stokes equation is a hot topic in recent years. Tremendous efforts have been made to develop the numerical methods for the Stokes eigenvalues, such as, finite element method [2, 3, 8, 13, 14, 24], triangular spectral element method [6, 25], the discontinuous Galerkin method [10].

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Solving the Stokes eigenvalue problem also needs to consider the difficulties encountered by the source problem. It is quite subtle to choose the approximation space pair of the velocity and pressure to guarantee stability and convergence [4]. The inf-sup condition, which is also referred to as LBB (Ladyzhenskaya-Babuška-Brezzi) condition, plays a core role in the numerical discretization. One of the most popular finite element space is the Taylor-Hood $\mathbb{P}^{m+1}/\mathbb{P}^m$, $m \geq 1$ element pair in the mixed finite element [4, 26].

This article will employ the discontinuous Galerkin method by patch reconstruction to study the Stokes eigenvalue problem. The method is an efficient numerical method for solving partial differential equations, which was first introduced in [18] for the elliptic problems, and applied to many other model problems [17, 19, 22]. In [20], Li *et al.* developed the discontinuous Galerkin method by patch reconstruction for the Stokes problem using mixed formulations where the approximation space is the general reconstruction space. In [21], the locally divergence-free space by patch reconstruction is utilized with the super penalty scheme to cancel out the pressure term.

This work incorporates the mixed formulations with the locally divergence-free space by patch reconstruction to solve the Stokes eigenvalue problem. The reason for developing this new scheme is to overcome some drawbacks of previous results. Firstly, the commonly used element pair $\mathbb{P}^{m+1}/\mathbb{P}^m$ requires the reconstruction conducting independently to generate two approximation spaces. Secondly, there may have some stability problems, such as the pressure-dependent consistent issue [15], the mass conservation issue [9]. The locally divergence-free approximation space is preferred because it can avoid those problems by post-processing. But entirely abandoning the pressure term demands either constructing the global divergence-free approximation space or using the super penalty scheme. Constructing the global divergence basis requires the mesh geometry is simple. Meanwhile, the latter will lead to a large condition number of resulting linear systems. So we combine the mixed formulations and the patch reconstructed locally divergence-free approximation space to solve the Stokes eigenvalue problem. The locally divergence-free polynomial degrees of freedom (DOF) is much smaller than the general case. Therefore, the element patch needs to be chosen once while the element pair $\mathbb{P}^{m+1}/\mathbb{P}^m$ is employed. Moreover, the proposed method also enjoys the merits of the PRDG method. For instance, the high efficiency of utilizing the DOFs, the flexibility in mesh geometry.

The main contributions of this paper are summarized as follows.

- The commonly used element pair $\mathbb{P}^{m+1}/\mathbb{P}^m$ requires the reconstruction conducting independently to generate two approximation spaces.

- We combine the mixed formulations and the patch reconstructed locally divergence-free approximation space to solve the Stokes eigenvalue problem. The locally divergence-free polynomial degrees of freedom (DOF) is much smaller than the general case so we only chosen the element patch once while $\mathbb{P}^{m+1}/\mathbb{P}^m$ element pair.