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A Variant Modified Skew-Normal Splitting Iterative Method for Non-Hermitian Positive Definite Linear Systems

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Abstract. We propose a variant modified skew-normal splitting iterative method to solve a class of large sparse non-Hermitian positive definite linear systems. Applying the preconditioning technique we also construct the preconditioned version of the proposed method. Theoretical analysis shows that the proposed method is unconditionally convergent even when the real part and the imaginary part of the coefficient matrix are non-symmetric. Meanwhile, when the real part and the imaginary part of the coefficient matrix are symmetric positive definite, we prove that the preconditioned variant modified skew-normal splitting iterative method will also unconditionally converge. Numerical experiments are presented to illustrate the efficiency of the proposed method and show better performance of it when compared with some other methods.

AMS subject classifications: 65F10, 65F15, 65T10

Key words: Non-Hermitian matrix, skew-normal splitting, precondition, complex linear system.

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1. Introduction

In this paper, we consider to solve the following nonsingular complex linear system:

$$Ax \equiv (W + \mathrm{i}T)x = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian matrix, $b \in \mathbb{C}^n$, $\mathrm{i} = \sqrt{-1}$ is an imaginary unit and W, T are two $n \times n$ non-symmetric real matrices. If W, T are both symmetric, the system (1.1) is called a complex symmetric linear system. System (1.1)

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arises widely from many applications in the area of scientific computing, including the optical tomography in medical imaging [1], structural dynamics [20], lattice quantum chromodynamics [21], quantum mechanics [29], molecular scattering [27], electromagnetism [23], electrical power modeling [25], eddy current problem [4] and so on. More applications can be found in [18] and the references therein.

Many authors have contributed to the development of iterative methods for solving linear system (1.1) based on different splittings of matrix A. For example, we can uniquely split A as $A = H - S$, where

$$H = \frac{A + A^*}{2}, \quad S = \frac{A^* - A}{2}$$

are the Hermitian and skew-Hermitian parts of matrix A, respectively. Here, we use the superscript '$*$' to denote the conjugate transpose of a complex matrix, while t to denote the transpose of a real matrix in the afterwards discussion. Then, the Hermitian and skew-Hermitian splitting (HSS) iterative scheme proposed in [13] can be stated as

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I + S)x^{(k)} + b, \\ (\alpha I - S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.2}$$

Hereafter, α is always assumed to be a given positive constant and I the identity matrix with a suitable dimension. More generally, we can also split A to be $A = N - \tilde{S}$ or $A = P - \bar{S}$, where N is a normal matrix, P is a positive definite matrix, and $\tilde{S}, \bar{S}$ are skew-Hermitian matrices. It is easy to see that these two splittings of A are not unique. The normal skew-Hermitian splitting iterative method and the positive-definite skew-Hermitian splitting iterative method are studied in [12,14], respectively. See [2] for an overview. Let V be an $n \times n$ real symmetric positive definite matrix, the preconditioned HSS iteration (PHSS) method was discussed in [11,16,19], and the iterative scheme for PHSS method is

$$\begin{cases} (\alpha V + H)x^{(k+\frac{1}{2})} = (\alpha V + S)x^{(k)} + b, \\ (\alpha V - S)x^{(k+1)} = (\alpha V - H)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.3}$$

Some other developments of the Hermitian and skew-Hermitian splitting method can be found in [3,9,10,15,17,24,26,28,30–33].

Specially, when W, T are both real symmetric matrices, we have $H = W,\ S = -\mathrm{i}T$. Applying the HSS iterative scheme (1.2) to solve the complex symmetric linear system, we get

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \mathrm{i}T)x^{(k)} + b, \\ (\alpha I + \mathrm{i}T)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b. \end{cases} \tag{1.4}$$

In order to avoid complex computation at the second step of iteration (1.4), Bai *et al.* [6] constructed the following modified HSS (MHSS) iterative scheme to solve the complex symmetric linear system:

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \mathrm{i}T)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + \mathrm{i}W)x^{(k+\frac{1}{2})} - \mathrm{i}b. \end{cases} \tag{1.5}$$