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A Posteriori Error Analysis of a P_2-CDG Space-Time Finite Element Method for the Wave Equation

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Abstract. This paper develops a posteriori error bound for a space-time finite element method for the linear wave equation. The standard P_l conforming element is used for the spatial discretization and a P_2-CDG method is applied for the time discretization. The essential ingredients in the a posteriori error analysis are the twice time reconstruction functions and the $C^1(J)$-smooth elliptic reconstruction, which lead to reliable a posteriori error bound in view of the energy method. As an outcome, a time adaptive algorithm is proposed with the error equidistribution strategy. Numerical experiments are reported to illustrate the performance of the a posteriori error bound and the validity of the adaptive algorithm.

AMS subject classifications: 65M60, 65M15

Key words: P_2-CDG, wave equations, a posteriori error estimate.

1. Introduction

Consider the following initial boundary problem for the second order wave equation

$$\begin{cases} u_{tt} - \nabla \cdot \big(p(x)\nabla u\big) = f(x,t) & \text{in } \Omega \times J, \\ u(x,t) = 0 & \text{on } \Gamma \times J, \\ u(x,t) = u_0(x) & \text{on } \Omega \times \{0\}, \\ u_t(x,t) = u_1(x) & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

Here, the symbol J stands for the time interval $(0,T]$ for a given constant $T > 0$, Ω

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is a bounded polygonal/polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary Γ, $(\cdot)_t$ and $(\cdot)_{tt}$ denote respectively the first and second order derivatives in time. The scalar-value function $p(x) \in C(\bar{\Omega})$ satisfies $0 < \alpha_{\min} \leq p(x) \leq \alpha_{\max}$ for all $x \in \Omega$, where $\alpha_{\min}$ and $\alpha_{\max}$ are two positive constants. In addition, assume that $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. This problem can be used to describe the motion of a membrane and it can also be viewed as a prototype model for more sophisticated wave phenomena of practical interest, such as the propagation of elastic waves, electromagnetic radiation, and so on. Generally speaking, problem (1.1) does not have a closed-form solution, so there have developed many numerical methods to find its solution for practical applications, among which the space-time finite element method is a popular choice (cf. [2, 4, 8, 11–13, 25]).

From the perspectives of theoretic analysis and numerical simulation, the above methods with uniform partitions both in time and space direction are shown to be convergent. However, for the problems with rapid-change solutions, we often require to use very small mesh sizes uniformly to obtain numerical solutions with desired accuracy, leading to heavy computational overheads. The adaptive finite element method (AFEM) seems to be an effective approach to overcoming such difficulty. In order to produce an AFEM, we usually have to develop a posteriori error analysis for the underlying finite element method in advance. For the first order evolution problem (e.g. the parabolic problem), such studies are rather extensive and through; we refer to [5, 6, 19, 21, 23, 24] and the references therein for details.

However, to the best of our knowledge, there are few results about a posteriori error analysis for the second order evolution problem. In [9], Georgonlis *et al.* applied the technique of elliptic reconstruction used for parabolic problems in [21] to the wave equation and derived a posteriori error bound in the $L^\infty(L^2)$ norm. The a posteriori error analysis in [9] was very tricky, and the ideas are extended to the a posteriori error estimates of various space-time finite element methods for many mathematical physical problems, such as the interface problem [7], the elasticity problem [20], and the linear Schrödinger equation [17]. We remark that for the main results (Lemma 5.1 and Theorem 5.4) given in [9], the generic constants depend on the maximum of the rates of two neighbouring time step-sizes. So such estimates cannot be used directly to produce time adaptive methods using the error equidistribution strategy (cf. [6, 22]). In [14], Guo *et al.* first constructed a space-time method for (1.1), with the spatial discretization carried out by the standard conforming finite element method while the temporal discretization done by the linear continuous discontinuous Galerkin method (CDG) (cf. [18]). Then, they developed a posteriori error bound with the generic constants independent of the step sizes in time, from which two adaptive algorithms were devised for adaptive computation. It deserves to point out that this method is just 1st-order accurate in time.

On the other hand, there also exist some investigations on a posteriori error analysis of higher order time reconstruction for abstract second-order evolution equations. For example, Huang *et al.* [15] proposed a P_2-CDG method for abstract second-order evolution problems, and obtained a posteriori error estimation as well as a posteriori