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A Novel Class of Energy-Preserving Runge-Kutta Methods for the Korteweg-de Vries Equation

Yue Chen[1,2], Yuezheng Gong[1,2,3,*], Qi Hong[1,2,3] and Chunwu Wang[1,2]

[1] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China*
[2] *Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 211106, China*
[3] *Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing 210023, China*

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Abstract. In this paper, we present a quadratic auxiliary variable approach to develop a new class of energy-preserving Runge-Kutta methods for the Korteweg-de Vries equation. The quadratic auxiliary variable approach is first proposed to reformulate the original model into an equivalent system, which transforms the energy conservation law of the Korteweg-de Vries equation into two quadratic invariants of the reformulated system. Then the symplectic Runge-Kutta methods are directly employed for the reformulated model to arrive at a new kind of time semi-discrete schemes for the original problem. Under consistent initial conditions, the proposed methods are rigorously proved to maintain the original energy conservation law of the Korteweg-de Vries equation. In addition, the Fourier pseudo-spectral method is used for spatial discretization, resulting in fully discrete energy-preserving schemes. To implement the proposed methods effectively, we present a very efficient iterative technique, which not only greatly saves the calculation cost, but also achieves the purpose of practically preserving structure. Ample numerical results are addressed to confirm the expected order of accuracy, conservative property and efficiency of the proposed algorithms.

AMS subject classifications: 65M06, 65M70

Key words: Quadratic auxiliary variable approach, symplectic Runge-Kutta scheme, energy-preserving algorithm, Fourier pseudo-spectral method.

1. Introduction

In this paper, we are concerned with the Korteweg-de Vries (KdV) equation

*Corresponding author. *Email address:* gongyuezheng@nuaa.edu.cn (Y. Gong)

$$u_t + \eta u u_x + \mu^2 u_{xxx} = 0, \quad (x,t) \in [a,b] \times (0,T] \tag{1.1}$$

with periodic boundary condition

$$u(a,t) = u(b,t), \quad t \in [0,T], \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{1.3}$$

where η and μ are two real parameters. It is an important nonlinear hyperbolic equation with smooth solution at all times and also a mathematical waves on shallow water surfaces. Eq. (1.1) has been used to describe various phenomena such as waves in bubble-liquid mixtures, acoustic waves in an anharmonic crystal, magnetohydrodynamic waves in warm plasma and ion acoustic waves [50].

In the past half century, numerous numerical methods have been developed for the KdV equation, including spatially numerical approximations such as Galerkin methods [4, 46–48], finite difference schemes [3, 52], Fourier spectral or pseudo-spectral methods [7, 22], etc., and operator splitting and exponential-type integrators [28, 29] for the temporal discretization and so on. Recently, there has been a surge on constructing numerical methods for dynamical systems governed by differential equations to preserve as many properties of the continuous system as possible. Numerical methods that preserve at least some of the structural properties of the continuous dynamical system are called geometric integrators or structure-preserving algorithms [20, 27]. Many geometric integrators have been presented for the KdV equation, especially the symplectic and multisymplectic schemes [3, 30, 44, 52]. In recent years, various energy-preserving and momentum-preserving algorithms have been developed for this equation as well [7, 18, 19, 35]. More recently, some local structure-preserving algorithms, originally discussed by Wang *et al.* [45], have been applied for the KdV equation [22, 43]. However, most of the existing structure-preserving algorithms are only up to second order in time, which cannot usually provide long time accurate solutions with a given large time step.

As a matter of fact, how to devise high-order invariant-preserving methods for conservative systems has attracted a lot of attention in recent years. It is well known that all Runge-Kutta (RK) methods preserve linear invariants, while only those that satisfy the symplectic condition conserve all quadratic invariants [17, 38]. For canonical Hamiltonian systems, many high-order energy-preserving algorithms have been developed, including high-order averaged vector field (AVF) methods [36, 37, 42], Hamiltonian Boundary Value Methods (HBVMs) [10], energy-preserving variant of collocation methods [26], and time finite element methods [41]. In addition, the above mentioned high-order energy-preserving methods are also valid for Hamiltonian systems with constant skew-symmetric structural matrix. For general conservative systems, these methods should be further discussed (e.g., see [5, 12, 16]). As far as we know, the HBVMs have been applied for the KdV equation to obtain high-order energy-preserving methods [6, 7, 40]. It should be noted that all of these methods involve integrals, which often