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Two-Phase Image Segmentation by the Allen-Cahn Equation and a Nonlocal Edge Detection Operator

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Abstract. Based on a nonlocal Laplacian operator, a novel edge detection method of the grayscale image is proposed in this paper. This operator utilizes the information of neighbor pixels for a given pixel to obtain effective and delicate edge detection. The nonlocal edge detection method is used as an initialization for solving the Allen-Cahn equation to achieve two-phase segmentation of the grayscale image. Efficient exponential time differencing (ETD) solvers are employed in the time integration, and finite difference method is adopted in space discretization. The maximum bound principle and energy stability of the proposed numerical schemes are proved. The capability of our segmentation method has been verified in numerical experiments for different types of grayscale images.

AMS subject classifications: 68U10, 65K10, 65M12, 62H35

Key words: Image segmentation, Allen-Cahn equation, nonlocal edge detection operator, maximum principle, energy stability.

1. Introduction

Image segmentation is the process of dividing an image domain into some disjoint regions according to a characterization of the image within or in-between the regions [25]. The characterization can be color, shape, edge, texture, or any feature that can distinguish each region from others. Image segmentation is involved in many fields, such as remote sensing, medical image recognition, robotics, and visual field monitoring.

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Among many segmentation methods, variational methods have attracted considerable attention. A typical variational method for image segmentation is based on minimizing an objective energy functional, for instance, the Mumford-Shah (MS) model [26],

$$E_{MS}(u,I)=\int_{\Omega\backslash\Gamma}|\nabla u|^2dx+\mu\,\mathrm{Length}(\Gamma)+\lambda\int_{\Omega}(u-I)^2dx, \tag{1.1}$$

which is associated with the partition determined by original image I and a union of closed edges Γ. The function u in the MS model is a piecewise smooth approximation to I and μ,λ are positive constants. Under many circumstances, the function u can degenerate to piecewise constant [7] and the following model can be achieved:

$$E_{CV}(D,C_1,C_2)=\mathrm{Per}(D;\Omega)+\lambda_1\int_D(C_1-I)^2dx+\lambda_2\int_{\Omega\backslash D}(C_2-I)^2dx, \tag{1.2}$$

which is called the two-phase piecewise constant MS model, also known as Chan-Vese model considered previously by Chan and Vese with level set formulation in [7, 30]. Here, $Per(D;\Omega)$ denotes the perimeter of the closed curve between two regions D and $\Omega\backslash D$, λ_1 and λ_2 are positive constants, and C_1,C_2 are the average intensities of two phases respectively, which change along with the image intensity. Generally, the objective energy functional contained two parts, including image and partition constraints. Image constraints yield functional terms called fitting terms, which measure how close an approximation fits the original image. Partition constraints give rise to partition priors which usually describe a geometric aspect of the edges, such as their length and smoothness. These two constraints have a wide variety, based on the region [19,20,34] or edge [5, 18].

Different approximations combined with different constraints result in various energy functionals. To solve these models for image segmentation, the level set method, phase-field method, and threshold dynamic method have been successfully used. Wherein, level set methods were proposed by Osher and Sethian in [27, 28]. The closed curves describing a segmentation region can split or merge flexibly to implement segmentation with complex structures. Subsequently, it has been found that the energy functional can be replaced by another form and derived the variational level set method [19, 25], which has advantages on numerical stability. Inspired by the variational level set method, various phase-field models have been derived for image segmentation through the Γ-convergence [1] for the length term of partition constraints, see e.g., [2, 17, 21, 33] and references therein. Merriman *et al.* (MBO) [24] developed a threshold dynamic method for the motion of an interface driven by the mean curvature. As pointed in [14], the length term of partition constraints can be estimated by the convolution of a heat kernel. As a consequence, a kind of threshold dynamic methods [15, 31, 32] had been constructed and obtained effective segmentation results.

In this paper, we plan to adopt a phase-field approach of the Chan-Vese model (1.2) for the two-phase segmentation of grayscale images. An alternating minimization method will be used to update the phase variable u, the two-phase average intensities