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Efficient and Accurate Numerical Methods Using the Accelerated Spectral Deferred Correction for Solving Fractional Differential Equations

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Abstract. We develop an efficient and accurate spectral deferred correction (SDC) method for fractional differential equations (FDEs) by extending the algorithm in [14] for classical ordinary differential equations (ODEs). Specifically, we discretize the resulted Picard integral equation by the SDC method and accelerate the convergence of the SDC iteration by using the generalized minimal residual algorithm (GMRES). We first derive the correction matrix of the SDC method for FDEs and analyze the convergence region of the SDC method. We then present several numerical examples for stiff and non-stiff FDEs including fractional linear and nonlinear ODEs as well as fractional phase field models, demonstrating that the accelerated SDC method is much more efficient than the original SDC method, especially for stiff problems. Furthermore, we resolve the issue of low accuracy arising from the singularity of the solutions by using a geometric mesh, leading to highly accurate solutions compared to uniform mesh solutions at almost the same computational cost. Moreover, for long-time integration of FDEs, using the geometric mesh leads to great computational savings as the total number of degrees of freedom required is relatively small.

AMS subject classifications: 65N35, 65E05, 65M70, 41A05, 41A10, 41A25

Key words: Stiff problem, generalized minimal residual, geometric mesh refinement, long time evolution, fractional phase field models.

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1. Introduction

Fractional differential equations (FDEs) have been effective in modeling anomalous diffusion as well as capturing long-range spatio-temporal interactions [11, 24, 26]. Analytic solutions of some FDEs are typically obtained by using special functions (e.g., Wright functions) for simple linear problems [22]. However, it is usually difficult to obtain analytic solutions for more complex FDEs, especially for nonlinear problems. For time-fractional differential equations, high-order numerical methods including the finite difference method [4,10,18,19,29] and the polynomial based spectral method [17] have been developed for smooth solutions. However, due to the singular kernel of the fractional operator, the solutions of FDEs are usually of low regularity in the usual Sobolev space. To resolve this issue, Jin *et al.* used corrections to restore the theoretical high-order convergence rate [15] (see also [20, 21]). A different approach developed by Zayernouri and Karniadakis [34] was to use the poly-fractonomials as basis functions, which match the singularity of the kernel, giving spectral accuracy for a smooth source term; a rigorous error analysis was established in [6]. Nevertheless, this method cannot be extended to more general FDEs. Moreover, none of the aforementioned methods can easily handle the long-time evolution since it is computationally expensive and has high memory requirement due to the non-locality of the fractional operator. A number of papers using the finite difference method have been published to address this issue by using the FFT based discrete convolution or representing the discrete weights into integral forms [12, 13, 32, 33, 35]. However, spectral methods, especially multi-domain spectral methods are more favorable for fractional problems, giving much higher accuracy than local methods [5, 16].

In this work, we aim at developing an efficient and accurate numerical scheme using the spectral deferred correction (SDC) method for FDEs. The SDC method was first introduced by Dutt *et al.* in [9] to construct high-order stable methods for solving ordinary differential equations (ODEs). Some early work on using the SDC method for nonlocal equations or FDEs can be found in [3, 23, 25, 31]. In these works, again, the low regularity of the solutions was not addressed explicitly. It was shown that the convergence rate is $\mathcal{O}(\Delta T^{(2-\alpha)(k+1)})$ (or $\mathcal{O}(\Delta T^{(2-\alpha)+k})$) for the uniform mesh (or the Gauss-Lobatto mesh) in [23] while the convergence rate is $\mathcal{O}(\Delta T^{\min(p+1+\alpha,\alpha(k+1)+\delta)}), \delta = 1$ or 2 in [3], where ΔT is the length of the subdomain, α is the order of the time fractional operator, k is the number of SDC iterations and p is the degree of the polynomial used for the SDC scheme. More details about these parameters will be given in the next section. This means that there is an increase of the order of $2-\alpha, 1$ or α for each iteration for the global error. However, this fails and the so called order reduction occurs when solving stiff problems. Moreover, it is not computationally efficient to solve large systems by using the SDC methods developed in these works.

In the current work, we extend the techniques in [14] (see also [27]) for classical ODEs to fractional ODEs. In particular, we use the SDC method to discretize the time fractional operator for the fractional initial and/or boundary problems, and accelerate the convergence of the SDC iteration with the generalized minimal residual (GMRES)