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A Second-Order Semi-Implicit Method for the Inertial Landau-Lifshitz-Gilbert Equation

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Abstract. Electron spins in magnetic materials have preferred orientations collectively and generate the macroscopic magnetization. Its dynamics spans over a wide range of timescales from femtosecond to picosecond, and then to nanosecond. The Landau-Lifshitz-Gilbert (LLG) equation has been widely used in micromagnetics simulations over decades. Recent theoretical and experimental advances have shown that the inertia of magnetization emerges at sub-picosecond timescales and contributes significantly to the ultrafast magnetization dynamics, which cannot be captured intrinsically by the LLG equation. Therefore, as a generalization, the inertial LLG (iLLG) equation is proposed to model the ultrafast magnetization dynamics. Mathematically, the LLG equation is a nonlinear system of parabolic type with (possible) degeneracy. However, the iLLG equation is a nonlinear system of mixed hyperbolic-parabolic type with degeneracy, and exhibits more complicated structures. It behaves as a hyperbolic system at sub-picosecond timescales, while behaves as a parabolic system at larger timescales spanning from picosecond to nanosecond. Such hybrid behaviors impose additional difficulties on designing efficient numerical methods for the iLLG equation. In this work, we propose a second-order semi-implicit scheme to solve the iLLG equation. The second-order temporal derivative of magnetization is approximated by the standard centered difference scheme, and the first-order temporal derivative is approximated by the midpoint scheme involving three time steps. The nonlinear terms are treated semi-implicitly using one-sided interpolation with second-order accuracy. At each time step, the unconditionally

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unique solvability of the unsymmetric linear system is proved with detailed discussions on the condition number. Numerically, the second-order accuracy of the proposed method in both time and space is verified. At sub-picosecond timescales, the inertial effect of ferromagnetics is observed in micromagnetics simulations, in consistency with the hyperbolic property of the iLLG model; at nanosecond timescales, the results of the iLLG model are in nice agreements with those of the LLG model, in consistency with the parabolic feature of the iLLG model.

AMS subject classifications: 35Q99, 65Z05, 65M06

Key words: Inertial Landau-Lifshitz-Gilbert equation, semi-implicit scheme, second-order accuracy, micromagnetics simulations.

1. Introduction

Ferromagnetic materials are widely used for data storage devices due to the realization of fast magnetization dynamics under various external controls [4, 26]. In this scenario, the dissipative magnetization dynamics is mainly controlled by the magnetic degrees of freedom at timescales from picosecond (10^{-12} s) to nanosecond (10^{-9} s), which is typically modeled by the conventional Landau-Lifshitz-Gilbert (LLG) equation [10, 15]. However, some recent experiments including the observation of the spin dynamics at sub-picosecond timescales [2] as well as the realization of the magnetization reversal excited by the spin wave of sub-GHz frequency [11], indicated that ultrafast magnetic dynamics can be properly described by the LLG equation via adding an inertial term [3, 9, 18].

For the LLG equation with an inertial term, denoting τ as the characteristic timescale of the inertial effect, the magnetization dynamics can be roughly divided into two regimes: the diffusive regime at the timescale of $t \gg \tau$, and the hyperbolic regime at the timescale of $t \approx \tau$. In the hyperbolic regime, magnetization dynamics exhibits the inertial feature [17, 20]. From the modeling perspective, $\partial_t \mathbf{M}$ and $\mathbf{M} \times \partial_t \mathbf{M}$ control the time evolution of magnetization $\mathbf{M}(\mathbf{x}, t)$ in the LLG equation, and $\partial_{tt} \mathbf{M}$ is further added to account for the inertial effect. This modification leads to the inertial LLG (iLLG) equation [9, 18]. Mathematically, the LLG equation is a nonlinear system of equations of parabolic type with (possible) degeneracy. Under the condition $t \approx \tau$, the inertial term dominates and the iLLG equation is more like a nonlinear system of equations of hyperbolic type. While under the condition $t \gg \tau$, the inertial term can be ignored and the iLLG equation is more like a parabolic system. Therefore, a reliable numerical method for the iLLG equation should capture both the inertial dynamics at sub-picosecond timescales and the gyroscopic dynamics at nanosecond timescales.

There exist a large number of numerical methods for the LLG equation; see [8, 13] for reviews and references therein. First-order semi-implicit schemes such as the Gauss-Seidel projection method [16, 24] and the semi-implicit backward Euler method [7] are well established. And recently, a second order semi-implicit projection method with