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Application of Adapted-Bubbles to the Helmholtz Equation with Large Wavenumbers in 2D

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Abstract. An adapted-bubbles approach which is a modification of the residual-free bubbles (RFB) method, is proposed for the Helmholtz problem in 2D. A new two-level finite element method is introduced for the approximations of the bubble functions. Unlike the other equations such as the advection-diffusion equation, RFB method when applied to the Helmholtz equation, does not depend on another stabilized method to obtain approximations to the solutions of the sub-problems. Adapted-bubbles (AB) are obtained by a simple modification of the sub-problems. This modification increases the accuracy of the numerical solution impressively. We provide numerical experiments with the AB method up to $ch = 5$ where c is the wavenumber and h is the mesh size. Numerical tests show that the AB method is better by far than higher order methods available in the literature.

AMS subject classifications: 65N30, 65N06

Key words: Helmholtz equation, adapted-bubbles, residual-free bubbles, two-level finite element.

1. Introduction

Enriching linear finite element space with residual-free bubble functions is a general framework for the discretizations of the partial differential equations such as the Helmholtz equation [12], advection-diffusion equation [3,14] and Navier-Stokes equation [20]. These functions strongly satisfy the original differential equations and hence obtaining the bubble functions is generally as difficult as solving the original problem such as the advection-diffusion equation [3]. Unlike it was stated in [13], we will show that this is not the case for the Helmholtz problem. Obtaining the bubble functions on triangular elements is easier than solving the original problem. The standard Galerkin finite element method can be used with a coarse mesh to obtain efficient approximations to the bubble functions.

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The residual-free bubbles method produces the exact solution of linear differential equations in the one-dimensional case. However, the method in higher-dimensions is approximate and as we will show for the Helmholtz problem in this article, its contribution to the stabilization of the standard Galerkin method is not well adjusted. We modify the residual-free bubbles (RFB) method in 2D by multiplying the right-hand side of the bubble equations with a constant. This operation impressively increases the accuracy. The new bubbles are no more residual-free and we call them adapted-bubbles (AB). We provide the optimal values of the constants for the triangular and rectangular elements separately. We apply a two-level finite element method using the standard linear Galerkin finite element method to get approximations to the bubble functions.

We provide analysis to show how the AB method mitigates the pollution error. To this end, we approximate the bubble functions with piece-wise defined linear functions so-called pseudo-bubbles. The analysis give rise to a fourth-order finite difference scheme with seven-point stencil for the plane waves. It is perfectly applicable on polygonal and triangular domains. We use this method to do comparison with the AB method.

Standard discretizations when applied to the Helmholtz problem suffer from the pollution effect when the wavenumber is large [2]. Moreover standard iterative solvers are ineffective in obtaining the solutions of the discrete Helmholtz equation [9]. There is a great effort in literature to overcome these difficulties. Among the discretization techniques, there are finite difference [10,24], finite element [1,18,27], discontinuous Galerkin [6,11], virtual element [21], and boundary element methods [19]. At the same time, there is a great effort to develop efficient preconditioners, such as multigrid [4,7,8,15,26] and domain decomposition methods [16,17,25].

The AB method proposed in this article works for very large ch. We provide numerical experiments up to $ch = 5$ but, in principle, it works for larger ch as long as the parameters are obtained with the Algorithm 1 proposed in this article. Numerical results show that the AB method is better by far than higher order methods available in the literature. It works in the regime ($ch > 3.5$) where the other methods do not work.

The rest of this paper is organized as follows. We review the RFB method for the Helmholtz equation in Section 2. We explain how to implement two-level finite element method in 1D and provide analysis to show the contribution of the bubble functions in reducing the pollution error in Section 3. Section 4 is devoted to the analysis of the RFB method in 2D. We propose the AB method for triangular elements in Section 5. The AB method is considered with rectangular elements in Section 6. We finish with concluding remarks in Section 7.

2. Residual-free bubbles method (RFB) for the Helmholtz equation

We start with considering the Helmholtz problem in 1D with Dirichlet boundary conditions on unit interval