Numer. Math. Theor. Meth. Appl. Vol. **16**, No. 2, pp. 489-510
doi: 10.4208/nmtma.OA-2022-0163 May 2023

Mixed Finite Element Methods for the Ferrofluid Model with Magnetization Paralleled to the Magnetic Field

Yongke Wu[1] and Xiaoping Xie[2,*]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China*
[2] *School of Mathematical, Sichuan University, Chengdu 610064, China*

Received 19 October 2022; Accepted (in revised version) 11 January 2023

Abstract. Mixed finite element methods are considered for a ferrofluid flow model with magnetization paralleled to the magnetic field. The ferrofluid model is a coupled system of the Maxwell equations and the incompressible Navier-Stokes equations. By skillfully introducing some new variables, the model is rewritten as several decoupled subsystems that can be solved independently. Mixed finite element formulations are given to discretize the decoupled systems with proper finite element spaces. Existence and uniqueness of the mixed finite element solutions are shown, and optimal order error estimates are obtained under some reasonable assumptions. Numerical experiments confirm the theoretical results.

AMS subject classifications: 65N55, 65F10, 65N22, 65N30

Key words: Ferrofluid flow, decoupled system, mixed finite element method, error estimate.

1. Introduction

Ferrofluids are colloidal liquids consisting of non-conductive nanoscale ferromagnetic or ferrimagnetic particles suspended in carrier fluids, and have extensive applications in many technology and biomedicine fields [23,31]. There are two main ferrohydrodynamics (FHD) models which treat ferrofluids as homogeneous monophase fluids: the Rosensweig's model [24,25] and the Shliomis' model [18,26]. The main difference between these two models lies in that the former one considers the internal rotation of the nanoparticles, while the latter deals with the rotation as a magnetic torque. We refer to [1–4,22] for some existence results of solutions to the two FHD models.

The FHD models are coupled nonlinear systems of the Maxwell equations and the incompressible Navier-Stokes equations. There are limited works in the literature on the numerical analysis of the FHD models. In [16,17,27,30], several numerical

*Corresponding author. *Email addresses:* `wuyongke1982@sina.com` (Y. Wu), `xpxie@scu.edu.cn` (X. Xie)

schemes were applied to solve reduced two dimensional FHD models where some nonlinear terms of the original models are dropped. Nochetto *et al.* [21] showed the energy stability of the Rosensweig's model, proposed an energy-stable numerical scheme using finite element methods, and gave the existence and convergence of the numerical solutions. Recently, Wu and Xie [29] developed a class of energy-preserving mixed finite element methods for the Shliomis' FHD model, and derived optimal error estimates for both the the semi- and fully discrete schemes. We also note that in [32] an unconditionally energy-stable fully discrete finite element method was presented for a two-phase FHD model.

In this paper, we consider the Shliomis' FHD model with the assumption that the magnetization field is parallel to the magnetic field. Under this assumption, the magnetization equation in the Shliomis' model degenerates to the Langevin magnetization law [17, 24, 25]. We introduce some new variables to transform the model into two main decoupled subsystems, i.e., a nonlinear elliptic equation and the incompressible Navier-Stokes equations. We apply proper finite element spaces to discretize the nonlinear decoupled systems, prove the existence and, under some reasonable assumptions, uniqueness of the finite element solutions, and derive optimal error estimates. We also show that our scheme preserves the ferrofluids' nonconductive property $\operatorname{curl} \boldsymbol{H} = 0$ exactly.

The rest of this paper is organized as follows. In Section 2, we introduce several Sobolev spaces, give the governing equation of the FHD model with magnetization paralleled to magnetic field, reform the FHD model equivalently, and construct the weak formulations. In Section 3, we recall the finite element spaces, show the existence and uniqueness of solutions for the constructed finite element methods, and give the optimal order error estimates. In Section 4, some numerical experiments will be given to verify our theoretical results.

2. Preliminary

In this section, we introduce several Sobolev spaces, give the governing equations of the FHD model with magnetization paralleled to magnetic field, derive the equivalent decoupled systems, and present the weak formulations.

2.1. Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded and simply connected convex domain with Lipschitz boundary $\partial\Omega$, and $\boldsymbol{n}$ be the unit outward normal vector on $\partial\Omega$.

For any $p \geq 1$, we denote by $L^p(\Omega)$ the space of all power-p integrable functions on Ω with norm $\|\cdot\|_{L^p}$. For any nonnegative integer m, denote by $H^m(\Omega)$ the usual m-th order Sobolev space with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$. In particular, $H^0(\Omega) = L^2(\Omega)$ denotes the space of square integrable functions on Ω, with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. For the vector spaces $\boldsymbol{H}^m(\Omega) := (H^m(\Omega))^d$ and $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^d$, we use the same notations of norm, semi-norm and inner product as those for the scalar cases.