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Analysis of Deep Ritz Methods for Semilinear Elliptic Equations

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Abstract. In this paper, we propose a method for solving semilinear elliptical equations using a ResNet with ReLU^2 activations. Firstly, we present a comprehensive formulation based on the penalized variational form of the elliptical equations. We then apply the Deep Ritz Method, which works for a wide range of equations. We obtain an upper bound on the errors between the acquired solutions and the true solutions in terms of the depth $\mathcal{D}$, width $\mathcal{W}$ of the ReLU^2 ResNet, and the number of training samples n. Our simulation results demonstrate that our method can effectively overcome the curse of dimensionality and validate the theoretical results.

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Key words: Semilinear elliptic equations, Deep Ritz method, ReLU^2 ResNet, convergence rate.

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1. Introduction

Solving semilinear partial differential equations in high dimensional space is a challenging problem in physics and engineering with applications in hydromechanics (Navier-Stokes equations, Burgers equations) [5,11,23], quantum mechanics (Gross Pitaevskii equations) [3], variational geometry (Plateaus equations) [13], and more. Traditional numerical methods such as finite element, finite difference, and finite volume encounter the curse of dimensionality, where the number of parameters exponentially increases as the dimension grows, rendering these mesh-based methods im-

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practical. Recent attempts have been made to overcome this challenge, with one of the most promising tools being deep neural networks (DNN). The approximability of DNNs has been shown to overcome the curse of dimensionality, leading to the development of related methods [16, 38, 39], such as physics-informed neural networks (PINNs) [10, 20, 21, 31–33], Deep Galerkin method (DGM) [9, 22, 26, 36], and weak adversarial networks (WAN) [2, 7, 40].

The Deep Ritz method (DRM) is one of the most renowned approaches in the field of elliptic equations, capable of solving both the equations and the eigenvalue problems [9, 12, 14, 17, 19, 25, 27, 28, 30]. In this article, we present its application in nonlinear elliptic equations and provide a convergent analysis. To apply the method, we identify the functional variation that corresponds to the PDEs, and then replace the trial function with a deep neural network (DNN). We subsequently discretize it using the Monte Carlo algorithm [18, 37] and solve the discretized variation to approximate the solution. By following these steps, we can divide the error into two components: the approximation error and the statistical error. To bound the statistical error, we need to calculate the infinity norm of both the solution and its derivative [14, 28]. However, this requirement narrows down the method's applicability. To address this issue, we can use one of two methods. The first is to restrict the feasible parameter region of DNN [27]. In this case, the statistical error can be easily estimated, as the Rademacher complexity can be computed in the parameter space. However, the approximation error can be challenging to compute, especially for DNN with large depth. The second method is the one we propose in this article. By directly bounding the $W^{1,\infty}$ norm of the neural networks, we estimate the Rademacher complexity in the function space, and we can obtain the approximation error through the traditional mollifier technique.

The outline of this paper is as follows. In Section 2, we establish the primary problem of our article and introduce the notation we use. In Section 3, we present the variational loss of the problem and construct a simple error decomposition. The main theorem of the article is presented in Section 4 before its proof, for ease of reading. In Section 5, we provide numerical results to verify the effectiveness of the proposed method. Finally, we conclude the main body of our article with a discussion in Section 6. In Appendix A, we provide some lengthy proofs of the lemma in Section 4.3.

2. Preliminaries and notations

In this article, we consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + f(u) = g & \text{in } \Omega, \\ u + \dfrac{1}{\varepsilon}\dfrac{\partial u}{\partial n} = h & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\varepsilon \in (0, +\infty]$. The interval for ε includes the cases of Dirichlet boundary condition ($\varepsilon = +\infty$) and Robin boundary condition ($\varepsilon \in (0, +\infty)$). We limit our equation to the following assumption.