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A Novel Iterative Method to Find the Moore-Penrose Inverse of a Tensor with Einstein Product

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Abstract. In this study, based on an iterative method to solve nonlinear equations, a third-order convergent iterative method to compute the Moore-Penrose inverse of a tensor with the Einstein product is presented and analyzed. Numerical comparisons of the proposed method with other methods show that the average number of iterations, number of the Einstein products, and CPU time of our method are considerably less than other methods. In some applications, partial and fractional differential equations that lead to sparse matrices are considered as prototypes. We use the iterates obtained by the method as a preconditioner, based on tensor form to solve the multilinear system $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$. Finally, several practical numerical examples are also given to display the accuracy and efficiency of the new method. The presented results show that the proposed method is very robust for obtaining the Moore-Penrose inverse of tensors.

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Key words: Tensor, iterative methods, Moore-Penrose inverse, Einstein product.

1. Introduction

Tensors occur in a wide variety of application areas such as document analysis, psychometrics, formulation an n-person noncooperative game, medical engineering, chemometrics, higher-order, and so on [5, 19–21, 26, 29, 34]. In this paper, we denote matrices with uppercase letters $A, B, \ldots$, and tensors are signified by calligraphic font $\mathcal{A}, \mathcal{B}, \ldots$. Suppose that N is a positive integer, and an N-th order tensor $\mathcal{A} = (a_{i_1 \ldots i_N})_{1 \leq i_j \leq P_j}$ is a multidimensional array with $P_1 \ldots P_N$ entries. The tensor $\mathcal{A}$ is called a hyper-matrix, or tensors are higher-order generalizations of vectors and matrices. Let $\mathbb{R}^{P_1 \times \cdots \times P_N}$ show the space of N-th order tensors.

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In the following, we give some definitions of tensors and the Einstein product.

Definition 1.1 ([6]). *Let N and M be positive integers, also $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1 \times \cdots \times Q_N \times K_1 \times \cdots \times K_M}$. Then the Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:*

$$(\mathcal{A} *_N \mathcal{B})_{p_1 \ldots p_N k_1 \ldots k_M} = \sum_{q_N}^{Q_N} \cdots \sum_{q_1}^{Q_1} a_{p_1 \ldots p_N q_1 \ldots q_N} b_{q_1 \ldots q_N k_1 \ldots k_M}, \tag{1.1}$$

therefore, $\mathcal{A} *_N \mathcal{B} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times K_1 \times \cdots \times K_M}$.

Note that if $N = M = 1$, the Einstein product reduces to the standard matrix multiplication.

Definition 1.2. *Inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$ is defined as follows:*

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{q_N=1}^{Q_N} \cdots \sum_{q_1=1}^{Q_1} \sum_{p_N=1}^{P_N} \cdots \sum_{p_1=1}^{P_1} x_{p_1 \cdots p_N q_1 \cdots q_N} y_{q_1 \cdots q_N p_1 \cdots p_N}.$$

Definition 1.3 ([6]). *Let $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$ be a tensor, then transpose and Frobenius norm of the tensor $\mathcal{A}$ are defined as follows:*

$$(\mathcal{A}^T)_{p_1 \ldots p_N q_1 \ldots q_N} = (\mathcal{A})_{q_1 \ldots q_N p_1 \ldots p_N},$$

and

$$\|\mathcal{A}\| = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle} = \sqrt{\sum_{q_N=1}^{Q_N} \cdots \sum_{q_1=1}^{Q_1} \sum_{p_N=1}^{P_N} \cdots \sum_{p_1=1}^{P_1} |a_{q_1 \ldots q_N p_1 \ldots p_N}|^2},$$

respectively.

Definition 1.4 ([6]). *A tensor $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times P_1 \times \cdots \times P_N}$ is called diagonal if for all $p_l \neq q_l$, $l = 1, \ldots, N$ we have $a_{p_1 \ldots p_N q_1 \ldots q_N} = 0$. A diagonal tensor $\mathcal{I} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times P_1 \times \cdots \times P_N}$ is identity if $i_{p_1 \ldots p_N q_1 \ldots q_N} = \Pi_{l=1}^{N} \delta_{p_l q_l}$, where*

$$\delta_{p_l q_l} = \begin{cases} 1, & p_l = q_l, \\ 0, & p_l \neq q_l. \end{cases}$$

Definition 1.5. *Suppose that $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times P_1 \times \cdots \times P_N}$, then $\mathcal{B} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times P_1 \times \cdots \times P_N}$ is said inverse of $\mathcal{A}$ with the Einstein product if*

$$\mathcal{A} *_N \mathcal{B} = \mathcal{I},$$

therefore $\mathcal{A}^{-1} = \mathcal{B}$.

Proposition 1.1 ([38]). *If $\mathcal{A} \in \mathbb{R}^{P_1 \times \cdots \times P_N \times Q_1 \times \cdots \times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1 \times \cdots \times Q_N \times K_1 \times \cdots \times K_M}$, then*

$$(\mathcal{A} *_N \mathcal{B})^T = \mathcal{B}^T *_N \mathcal{A}^T, \quad \mathcal{I}_N *_N \mathcal{B} = \mathcal{B}, \quad \mathcal{B} *_M \mathcal{I}_M = \mathcal{B},$$

where $\mathcal{I}_N \in \mathbb{R}^{Q_1 \times \cdots \times Q_N \times Q_1 \times \cdots \times Q_N}$ and $\mathcal{I}_M \in \mathbb{R}^{K_1 \times \cdots \times K_M \times K_1 \times \cdots \times K_M}$, are identity tensors.