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Local MFS Matrix Decomposition Algorithms for Elliptic BVPs in Annuli

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Abstract. We apply the local method of fundamental solutions (LMFS) to boundary value problems (BVPs) for the Laplace and homogeneous biharmonic equations in annuli. By appropriately choosing the collocation points, the LMFS discretization yields sparse block circulant system matrices. As a result, matrix decomposition algorithms (MDAs) and fast Fourier transforms (FFTs) can be used for the solution of the systems resulting in considerable savings in both computational time and storage requirements. The accuracy of the method and its ability to solve large scale problems are demonstrated by applying it to several numerical experiments.

AMS subject classifications: Primary 65N35; Secondary 65N22

Key words: Local method of fundamental solutions, Poisson equation, biharmonic equation, matrix decomposition algorithms, fast Fourier transforms.

1. Introduction

The method of fundamental solutions (MFS) was introduced as a numerical method several decades ago [9, 28] and is by now an established method for accurately and effectively solving certain elliptic boundary value problems (BVPs) [7, 10, 14]. It is a boundary meshless method applicable to BVPs in which the fundamental solutions of the operators of the governing partial differential equations (PDEs) are known. The main attraction of the MFS is its simplicity since only the location of the boundary nodes (and corresponding sources) is required and neither interior discretization nor boundary integration are needed. As such, the MFS can be used to solve many complicated

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problems in science and engineering. However, identifying the source location (outside the domain) to yield optimal accuracy remains a challenge. In particular, when the domain is multi-connected containing several small cavities, placing the source points appropriately is a far from trivial task. It should be noted that the geometric shape of the domain and the boundary conditions (BCs) can affect the optimal source location. In theory, there are infinite ways of selecting the positions of the source points and many studies have attempted to resolve this issue. Moreover, since the traditional MFS is a global method and not a local method, the matrix resulting from such a discretization is full and poorly-conditioned. For large-scale problems requiring a large number of collocation points, implementationally demanding techniques such as domain decomposition or the fast multipole method need to be employed, which defeats the purpose of using the MFS in the first place since, as stated earlier, its major attraction is its simplicity. The above issues are, to a large extent, overcome in a recently developed local version of the MFS which yields sparse matrices. More specifically, the local method of fundamental solutions (LMFS) was introduced in the key paper [11] and has since been applied for the solution of a large variety of problems [4–6, 12, 15–19, 23–26, 29–31, 33–36].

The LMFS combines the traditional MFS with the ideas developed in other local meshless methods. As a result, in the implementation of the LMFS, both boundary and interior notes are required and, technically speaking, it may no longer be classified as a boundary method. Hence, we can consider the LMFS as a hybrid method combining features from both boundary and domain discretization methods. Fortunately however, the LMFS inherits the simplicity of the MFS.

The matrix resulting from the LMFS discretization is sparse which means that the corresponding linear systems can be efficiently solved. However, when the number of boundary and interior points becomes extremely large, the memory storage space and computational CPU time required become a problem.

In the current work we shall apply matrix decomposition algorithms (MDAs) [1] for the solution of Laplacian and biharmonic BVPs in annular domains. In MDAs, global systems (which could be sparse) are decomposed into many small systems, the solution of which results in substantial savings in both computational time and storage. Such algorithms have been employed extensively in MFS formulations, see e.g. [21]. MDAs have also been applied to radial basis function (RBF) collocation methods [27] and, in particular, to the local RBF method [2] which shares several features with the LMFS. In this paper, we apply MDAs to the LMFS to form an even more powerful meshless method which, as will be demonstrated, can readily handle problems with one million collocation nodes without the need of a high performance computer.

The types of BVPs examined in this study are presented in Section 2. In Section 3 we provide a detailed description of the LMFS for the solution of BVPs governed by the Laplace equation. The LMFS for biharmonic BVPs is presented in Section 4 while an alternative formulation for such problems is given in Section 5. In Section 7 we analyze the results obtained when the proposed method is applied to various test problems. Finally, we conclude with some comments and ideas about future work in Section 8.