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Enhancing RBF-FD Efficiency for Highly Non-Uniform Node Distributions via Adaptivity

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Abstract. Radial basis function generated finite-difference (RBF-FD) methods have recently gained popularity due to their flexibility with irregular node distributions. However, the convergence theories in the literature, when applied to nonuniform node distributions, require shrinking fill distance and do not take advantage of areas with high data density. Non-adaptive approach using same stencil size and degree of appended polynomial will have higher local accuracy at high density region, but has no effect on the overall order of convergence and could be a waste of computational power. This work proposes an adaptive RBF-FD method that utilizes the local data density to achieve a desirable order accuracy. By performing polynomial refinement and using adaptive stencil size based on data density, the adaptive RBF-FD method yields differentiation matrices with higher sparsity while achieving the same user-specified convergence order for nonuniform point distributions. This allows the method to better leverage regions with higher node density, maintaining both accuracy and efficiency compared to standard non-adaptive RBF-FD methods.

AMS subject classifications: 65N12, 65N35
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1. Introduction

In the past two decades, there has been important progress in developing adaptive mesh methods for PDEs. Mesh adaptivity is usually of two types in form: local mesh refinement and moving mesh method.

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Radial basis functions (RBFs) have been a popular choice in the development of kernel-based meshless methods for solving partial differential equations (PDEs) numerically. Besides collocation methods, the localized RBF-FD method has gained popularity in recent years due to its many advantages, including numerical stability on irregular node layouts, high computational speed and accuracy, easy local adaptive refinement, and excellent opportunities for large-scale parallel computing.

The idea of RBF and kernel-based differentiation can be traced back to [28], and it was formally introduced as the RBF-FD in [29]. Since then, a significant amount of research has been dedicated towards the robust development of the RBF-FD method [2, 3, 6, 9, 10, 17, 19, 20, 25, 30], as well as its application to various problems in science and engineering [4, 5, 7, 11–15, 26, 27]. In addition to the RBF-FD method, other collocation methods based on radial basis functions have also been proposed. For example, a global radial basis function collocation method in [21] was successfully applied to solve a computational fluid dynamic problem, and local RBF collocation methods were used to solve the diffusion problem in [22] and Hamiltonian PDEs in [32].

The RBF-FD method is advantageous since it works with scattered nodes, allowing for stencils with different configurations and overcoming the fixed grid/element limitation of conventional numerical methods. Unlike global meshless methods, RBF-FD computes weights locally using RBFs expanded at a fixed number of nearest nodes. Once weights at each node are computed, they can be stored and used for next-step computation, making weight computation a pre-processing step in solving time-dependent PDEs. Furthermore, weight computations at each node are independent processes, making RBF-FD a desirable method for parallel computing.

It has been demonstrated [2,6] that combining polynomial basis with polyharmonic spline radial basis functions (PHS+Poly) in the RBF-FD formulation leads to considerable improvements in robustness. Key benefits of the PHS+Poly approach include:

1. It is free of shape parameter, simplifying the formulation and eliminating the need for fine-tuning.

2. It is efficient compared to stable RBF-FD formulations based on infinitely smooth RBFs [20].

3. This method ensures accuracy near boundaries without ghost-nodes where stencils become highly one-sided [1].

4. It has the potential to maintain accuracy for large and sparse linear systems.

5. The convergence order depends mainly on the augmented polynomial degree, which determines the stencil size.

Existing convergence theories for scattered nodes require shrinking fill distance and fail to leverage regions of high node density. Numerically, non-adaptive methods using uniform stencil size and polynomial degree may exhibit loss of accuracy in low-density regions while adding unnecessary complexity in high-density regions,