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Analysis and Optimal Control of a System of Hemivariational Inequalities Arising in Non-Stationary Navier-Stokes Equation with Thermal Effects

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Abstract. In this paper, we primarily investigate the existence, dependence and optimal control results related to solutions for a system of hemivariational inequalities pertaining to a non-stationary Navier-Stokes equation coupled with an evolution equation of temperature field. The boundary conditions for both the velocity field and temperature field incorporate the generalized Clarke gradient. The existence and uniqueness of the weak solution are established by utilizing the Banach fixed point theorem in conjunction with certain results pertaining to hemivariational inequalities. The finite element method is used to discretize the system of hemivariational inequalities and error bounds are derived. Ultimately, a result confirming the existence of a solution to an optimal control problem for the system of hemivariational inequalities is elucidated.

AMS subject classifications: 65M15
Key words: Non-stationary Navier-Stokes equation, hemivariational inequalities, thermal effects, optimal control, existence and uniqueness.

1. Introduction

When characterizing various viscous fluid motion properties, such as the movement of clouds in the air, ripples and waves in water, etc., the Navier-Stokes (NS) equation is often used. As one of the most difficult equations in physics, the NS equation plays a very important role in engineering applications and has been widely used in various fields such as aerospace, automotive industry and shipbuilding industry. To describe

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different fluid problems, different boundary conditions need to be considered, which is a major important factor in determining fluid motion behavior.

Since the pioneering work of Fujita [9, 10], mathematical analysis and numerical methods for initial boundary value problems involving viscous incompressible fluids with nonlinear slip or leak boundary conditions have attracted considerable attention. These boundary conditions are non-smooth and sub-differentiable, and mathematically, variational inequalities can be used to effectively characterize such problems. Since then, variational inequalities have been widely used in fluid mechanics. However, as research continues to deepen, scientists have found that variational inequalities are associated with convex energy functionals. However, some fluid problems have more complex boundary conditions, such as fluid passing through a semipermeable membrane [23]. These non-smooth boundary conditions include non-monotonic relationships between physical quantities, thus, variational inequalities can not accurately characterize such problems. At this point, it is necessary to introduce Clarke subdifferential and employ hemivariational inequalities to describe them. Traditional fluid mechanics problems often assume the no-slip boundary condition, which is mathematically described using the Dirichlet boundary condition. However, with the detection of modern instruments such as atomic force microscopes and near-field laser velocimeters, it has been found that this no-slip boundary condition assumption does not hold in many scenarios, such as the slip phenomenon of blood flowing in blood vessels and the boundary slip phenomenon during gas flow.

Hemivariational inequality occupies an extremely important position in the study of various nonlinear problems in many fields such as chemistry, physics, biology, and engineering sciences. The study of variational inequalities originates from monotonicity theory and convexity theory, while hemivariational inequalities are mainly based on the Clarke subdifferential properties of locally Lipschitz functions and allow the inclusion of non-convex functions. Compared to variational inequalities, hemivariational inequalities have greater advantages in characterizing some practical problems. Moreover, thanks to the development of non-smooth analysis and multi-value analysis, the theory and numerical analysis of hemivariational inequality have developed rapidly in recent decades.

In recent years, many scholars have conducted in-depth research on the analysis for hemivariational inequalities in contact problems and published many important achievements [13, 16, 21, 26, 27]. Inspired by the theories of hemivariational inequalities arising in contact problems, in recent years, researchers have begun to focus on the hemivariational inequalities for fluid mechanics, and produced a series of papers [6, 12, 15, 17, 18]. The paper [6] focuses on the Stokes equations pertaining to steady flows of incompressible viscous fluids

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u} = \boldsymbol{0}, \\ u_n = 0, \ \ -\boldsymbol{\sigma}_\tau \in \partial j(\boldsymbol{u}_\tau). \end{cases}$$