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Penalized Schemes for Hamilton-Jacobi-Bellman Quasi-Variational Inequalities Arising in Regime Switching Utility Maximization with Optimal Stopping

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Abstract. The aim of this paper is to solve the Hamilton-Jacobi-Bellman (HJB) quasi-variational inequalities arising in regime switching utility maximization with optimal stopping. The HJB quasi-variational inequalities are penalized into the HJB equations and the convergence of the viscosity solution of the penalized HJB equations to that of the HJB variational inequalities is proved. The finite difference methods with iteration policy are used to solve the penalized HJB equations and the convergence is proved. The approach is implemented via numerical examples and the figures for the exercise boundaries and optimal strategies with sample paths are sketched.

AMS subject classifications: 65C20, 65C40, 65M06, 91G20, 91G60

Key words: Utility maximization, optimal stopping, stochastic control, regime switching, HJB variational inequalities, finite difference methods, iteration policy.

1. Introduction

The utility maximization is a kind of stochastic control problems. The dynamic programming approach is often applied to the optimal value function and the so-called HJB equation is derived (see the books Pham [34], Yong and Zhou [44] for the stochastic control and its applications). Since the HJB equation is a fully nonlinear PDE, the closed-form classical solution cannot be found except for some simple cases: a Black-Scholes complete market model with particular utility functions, see Bian *et al.* [8], Bian and Zheng [9]. For constrained market models it has to use numerical methods

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to solve the HJB equations. The standard approach to solve HJB equation by finite difference schemes is to discretize the derivatives in HJB equation and to solve the resulting finite dimensional control problem. The nonlinear discretized equations are often solved using policy iteration schemes (see e.g., [2, 3, 13, 15–19, 26, 27, 35, 36, 40, 41]). Among them, the work by Huang *et al.* [26, 27] and Babbin *et al.* [3] outlines the theory and implementation of the schemes for solving the coupled HJB equations arising in the American options under regime switching models. The convergence proofs are given therein.

A variant of utility maximization of terminal wealth is that investors may stop the investment before or at the maturity to achieve the overall maximum of the expected utility, which naturally leads to a mixed optimal control and stopping problem. The early work on this line includes Karatzas and Wang [29] and Dayanik and Karatzas [14] for properties of the value function at the initial time, Ceci and Bassan [12] for existence of viscosity solution of the variational equation, Henderson and Hobson [24] for equivalence of the value function in the presence of a Markov chain process and power utility. None of the above papers discusses the free boundary problem. Jian *et al.* [28] apply the dual transformation method to convert the nonlinear variational equation with power utility into an equivalent free boundary problem of a linear PDE and analyze qualitatively the properties of the free boundary and optimal strategies. The work is further extended in Guan *et al.* [21] to a problem with a call option type terminal payoff and power utility. Ma *et al.* [33] give rigorous analysis of the properties of the free boundary and construct the global approximation. It is well known that it is challenging to find the free boundary of a variational equation. The free boundary separates the exercise region from the continuation region and satisfies an integral equation which can be hardly solved. Finding the free boundary is much more difficult for the optimal investment stopping problem than for the American options pricing problem. The problems are often casted into a nonlinear quasi-variational inequality and the penalization methods are used to solve the nonlinear quasi-variational inequality. Witte and Reisinger [42] study the discrete quasi-variational inequalities arising from the discretization of an elliptic quasi-variational inequality using the penalty approach and Newton iterations. Azimzadeh *et al.* [1] study parabolic HJB quasi-variational inequalities, penalize it into a nonlinear HJB equation and use the policy iteration finite difference methods (FDMs) to solve the penalized HJB equations. Numerical implementations are not given in [1]. This paper extends the work of Azimzadeh *et al.* [1] to the regime-switching system of the HJB parabolic quasi-variational inequalities. Both the convergence analyses and the numerical implementations are given in this paper. Reisinger and Zhang [37] give the error estimates of penalty schemes for quasi-variational inequalities arising from impulse control problems. The setting of the problems is quite different from this paper, as the HJB operator in [37] is not time-dependent.

There has been active research in portfolio optimization with regime switching models. The regime switching model allows parameters of asset price dynamics to depend on a finite state Markov chain process. It provides good flexibility for charac-