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A Fast Compact Block-Centered Finite Difference Method on Graded Meshes for Time-Fractional Reaction-Diffusion Equations and Its Robust Analysis

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Abstract. In this article, an α-th ($0 < \alpha < 1$) order time-fractional reaction-diffusion equation with variably diffusion coefficient and initial weak singularity is considered. Combined with the fast $L1$ time-stepping method on graded temporal meshes, we develop and analyze a fourth-order compact block-centered finite difference (BCFD) method. By utilizing the discrete complementary convolution kernels and the α-robust fractional Grönwall inequality, we rigorously prove the α-robust unconditional stability of the developed fourth-order compact BCFD method whether for positive or negative reaction terms. Optimal sharp error estimates for both the primal variable and its flux are simultaneously derived and carefully analyzed. Finally, numerical examples are given to validate the efficiency and accuracy of the developed method.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Time-fractional reaction-diffusion equation, compact BCFD method, fast $L1$ method, α-robust unconditional stability, error estimates.

1. Introduction

Fractional differential equations have been widely used to describe challenging phenomena with long range time memory and spatial interactions due to their non-local nature [13, 30, 32, 45, 53, 57], and have drawn increasing attentions over the past several decades. In particular, time-fractional partial differential equations are typically

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used to model anomalous diffusion phenomenon. However, due to the nonlocal nature of fractional integral or differential operators, the analytical solutions are usually not available for such equations, and thus numerical modeling have been an efficient approach for studying the fractional differential models. So far, the time-fractional differential equations have been widely studied [6,19,21,27,32,33,43,50,54].

In this paper, we are interested in the following time-fractional reaction-diffusion problem:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha p(x,t) - \partial_x\big(a(x)\partial_x p(x,t)\big) + cp(x,t) = f(x,t), & (x,t) \in I \times (0,T_f], \\ p(x,0) = p^o(x), & x \in \bar{I} \end{cases} \tag{1.1}$$

under periodic boundary conditions, where $I := (x_l, x_r) \subset \mathbb{R}$ and $\bar{I} := I \cup \{x_l, x_r\}$. Moreover, the time-fractional derivative ${}_0^C\mathcal{D}_t^\alpha p$ in (1.1) is given in the Caputo sense [32]

$${}_0^C\mathcal{D}_t^\alpha p(x,t) := \int_0^t \omega_{1-\alpha}(t-s)\partial_s p(x,s)ds, \quad 0 < \alpha < 1,$$

where the kernel function $\omega_\beta(t) := t^{\beta-1}/\Gamma(\beta),\ t > 0$.

Throughout the paper, we suppose f and p^o are two given sufficiently smooth source and initial functions and the following assumptions hold [41,42]:

Assumption 1.1. Problem (1.1) has a unique solution $p(x,t)$, and there is a positive constant C_0 independent of α such that

$$\|p(\cdot,t)\|_{H^6} \le C_0, \tag{1.2}$$

and

$$\|\partial_t p(\cdot,t)\|_{H^5} \le C_0(1+t^{\alpha-1}), \quad \|\partial_{tt} p(\cdot,t)\|_{H^1} \le C_0(1+t^{\alpha-2}). \tag{1.3}$$

Assumption 1.2. Suppose that $a(x) \in C^1(\bar{I})$ is a periodic function, and there exist positive constants $a_* \le a^*$ such that $a_* \le a(x) \le a^*$. Besides, c is a constant that maybe positive or negative.

As pointed above, various methods have been presented to solve the time-fractional reaction-diffusion equation (1.1), see also [35,56]. However, the papers mentioned above only considered the case where c is non-negative, and most papers have ignored the possible presence of an initial layer in the typical solution near the initial time $t = 0$, and have presented convergence analysis under the unrealistic assumption $p(x,\cdot) \in C^2[0,T_f]$ or even high-order assumption, e.g. $C^3[0,T_f]$. It is pointed and proved in [38,42] that the typical solution of the α-th order time-fractional diffusion equation has weak singularity at $t = 0$, e.g. $\partial_t p \sim t^{\alpha-1}$. Thus, the forementioned theoretical analysis based on the assumption that the solution is smooth enough is not appropriate. To compensate for the weak singularity, an efficient strategy is to employ the graded meshes [4,20,44,49], that is concentrating more mesh points around the (weak) singular points to catch the rapid variation of the solution and use large stepsize