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Richardson Extrapolation of the Euler Scheme for Backward Stochastic Differential Equations

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Abstract. In this work, we consider Richardson extrapolation of the Euler scheme for backward stochastic differential equations (BSDEs). First, applying the Adomian decomposition to the nonlinear generator of BSDEs, we introduce a new system of BSDEs. Then we theoretically prove that the solution of the Euler scheme for BSDEs admits an asymptotic expansion, in which the coefficients in the expansions are the solutions of the system. Based on the expansion, we propose Richardson extrapolation algorithms for solving BSDEs. Finally, some numerical tests are carried out to verify our theoretical conclusions and to show the stability, efficiency and high accuracy of the algorithms.

AMS subject classifications: 65C30, 60H10, 60H35

Key words: Backward stochastic differential equations, Euler scheme, Adomian decomposition, Richardson extrapolation, asymptotic error expansion.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ being the natural filtration generated by a standard d_1-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^{d_1})^\top$, $0 \le t \le T$. We consider the following BSDE that is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \tag{1.1}$$

where T is a deterministic terminal time, $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^q$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times d_1} \longrightarrow \mathbb{R}^q$ are the terminal condition and the generator of the BSDE (1.1), respectively. Note that the stochastic integral with respect to W_t is of Itô's type, and X_t is a diffussion process. In this paper, we only consider the case where

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s, \quad 0 \le t \le T, \tag{1.2}$$

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where the functions $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d_1}$ are called the drift and the diffussion coefficients of the SDE (1.2). A pair of processes (Y_t, Z_t) is called an L^2-adapted solution of (1.1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies the BSDE (1.1).

In 1990, the existence and uniqueness of the solution of BSDEs are proved by Pardoux and Peng [27]. Since then, BSDEs becomes an important tool for formulating many problems in various important areas such as mathematical finance, stochastic optimal control, risk measure, game theory, and so on (see, e.g., [11, 25, 28, 31] and references therein).

As it is often difficult to solve BSDEs analytically, even for the linear BSDEs, numerical methods have become popular tools for solving BSDEs. In recent years, great efforts have been made for designing efficient numerical schemes for BSDEs and forward backward stochastic differential equations (FBSDEs). There are two main types of numerical schemes: the first one is based on numerical solution of a parabolic PDE which is related to a FBSDEs [10, 24], while the second type of schemes focus on discretizing FBSDEs directly [3, 5, 9, 17, 23, 32, 37]. From the temporal discretization viewpoint, popular strategies include Euler-type methods [14, 15, 35], θ-schemes [33, 39], Runge-Kutta schemes [8], multistep schemes [7, 13, 38, 40, 41], and strong stability preserving multistep (SSPM) schemes [12], to name a few. For fully coupled FBSDEs, there exists only few numerical studies and satisfactory results [26, 38]. We mention the work in [38], where a class of multistep type schemes are proposed, which turns out to be effective in obtaining highly accurate solutions of FBSDEs, and the work in [34], where the classical deferred correction (DC) method is adopted to design highly accurate numerical methods for fully coupled FBSDEs.

Our objective in this paper is to present a theoretical analysis for the Richardson extrapolation (RE) of the numerical solutions of the Euler scheme for BSDEs. It is well known that the extrapolation method, which was established by Richardson [30], is an efficient procedure for increasing the accuracy of approximations of many problems in numerical analysis. The effectiveness of this method relies heavily on the existence of an asymptotic expansion for the error of the numerical method which is used for the extrapolation procedure. The applications of RE to ordinary differential equations (ODEs) based on one-step schemes, e.g., Runge-Kutta methods are described, for example in [6, 16]. In addition, this method has been well demonstrated in its applications to the finite element and the mixed finite element methods for elliptic partial differential equations [4], Sobolev- and viscoelasticity-type equations [21], partial integro-differential equations [22], Fredholm and Volterra integral equations of the second kind [19], Volterra integro-differential equations [36], and to collocation methods in [20], etc..

The effectiveness and the high accuracy of Richardson extrapolation motivate us to use it to improve the accuracy of the solutions of the Euler scheme for BSDEs. To this end, we first theoretically prove that the solutions of the Euler scheme for BSDEs admit asymptotic expansions where the coefficients in the expansions satisfy a specific BSDEs system. To the best of our knowledge, such theoretical results are new in literature.