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Stability and Convergence of the Integral-Averaged Interpolation Operator Based on Q_1-Element in $\mathbb{R}^n$

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Abstract. In this paper, we propose an integral-averaged interpolation operator I_τ in a bounded domain $\Omega \subset \mathbb{R}^n$ by using Q_1-element. The interpolation coefficient is defined by the average integral value of the interpolation function u on the interval formed by the midpoints of the neighboring elements. The operator I_τ reduces the regularity requirement for the function u while maintaining standard convergence. Moreover, it possesses an important property of $\|I_\tau u\|_{0,\Omega} \leq \|u\|_{0,\Omega}$. We conduct stability analysis and error estimation for the operator I_τ. Finally, we present several numerical examples to test the efficiency and high accuracy of the operator.

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Key words: Integral-averaged interpolation operator, Q_1-element, stability, convergence.

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1. Introduction

The interpolation theory of functions is an important part of the theoretical analysis of the finite element method. It is used to estimate the value of the function at other points from the values at known data points [1–3, 6–8]. In practical applications, the selection of an appropriate interpolation operator is crucial to ensure computational accuracy and stability in the numerical methods for partial differential equations.

He and Feng [4] introduced a new basis function Q_1^c-element and used interval integration as one of the interpolation coefficients of the interpolation operator, which improved the convergence rate of the H^1-error of the interpolation function $u \in H^2(\Omega)$

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to the second order. However, the main focus of this article is on the properties of the basis function Q_1^c-element, and no analysis was conducted to determine whether average integration, as an interpolation coefficient, possesses superior properties.

In this study, we propose a new interpolation operator: the integral-averaged interpolation operator. This operator relaxes the regularity requirements for the interpolation function while preserving the convergence order. Assume that $\Omega = \prod_{i=1}^{n}(0, L_i) \subset \mathbb{R}^n$ is a bounded domain and $L_i \in (0, \infty)$ with $i = 1, \ldots, n$, while n is a positive integer. We consider J_i be a mesh of $(0, L_i)$ with $0 = x_{i,0} < x_{i,1} < \cdots < x_{i,l_i-1} < x_{i,l_i} = L_i$ and $\tau_i = L_i/l_i$, where l_i is an even positive integer with $i = 1, \ldots, n$. Also, we let $\tau = (\sum_{i=1}^{n} \tau_i^2)^{1/2}$. Then, we define the following finite element (FE) space basis on Q_1-element:

$$
\begin{aligned}
&X_{\tau_i} = \text{span}\{\phi_{i,k}(x_i),\ k = 0, \ldots, l_i\},\\
&X_\tau = X_{\tau_1} \times X_{\tau_2} \times \cdots \times X_{\tau_i} \times \cdots X_{\tau_n},
\end{aligned}
$$

where $\phi_{i,k}(x_i)$ is the linear basis function related to the nodal point $(x_{i,k})$ in $(0, L_i)$ satisfying

$$
\phi_{i,k}(x_{i,k}) = 1, \quad \phi_{i,k}(x_{i,l}) = 0 \quad \text{for} \quad k \neq l
$$

with $l = 0, \ldots, l_i$.

Now define the single-variable integral-averaged interpolation operator I_{τ_i}: $X \to X_{\tau_i}$ such that for each $u \in X = H_0^1(\Omega)$ as follows:

$$
I_{\tau_i} u = \sum_{k=0}^{l_i} \overline{u}(x_{i,k}) \phi_{i,k}(x_i),
$$

where

$$
\overline{u}(x_{i,k}) = \frac{1}{\tau_i} \int_{x_{i,k-\frac{1}{2}}}^{x_{i,k+\frac{1}{2}}} u(x_1, \ldots, x_i, \ldots, x_n) dx_i, \quad x_{i,k-\frac{1}{2}} = \frac{x_{i,k-1} + x_{i,k}}{2}
$$

with $k = 1, \ldots, l_i - 1$ and $\overline{u}(x_{i,k}) = u(x_1, \ldots, x_{i,k}, \ldots, x_n)$ for $k = 0, l_i$ and $i = 1, \ldots, n$. And, define the n-variable integral-averaged interpolation operator $I_\tau : X \to X_\tau$ such that for each $u \in X = H_0^1(\Omega)$ as follows:

$$
I_\tau u = I_{\tau_n} I_{\tau_{n-1}} \ldots I_{\tau_1} u.
$$

This paper presents a theoretical analysis of the stability and convergence of the single-variable integral-averaged interpolation operator and the n-variable integral-averaged interpolation operator. The analysis is conducted where the interpolation function $u \in H^m(\Omega) \cap H_0^1(\Omega)$, with $m = 1, 2$.

The paper is organized as follows. Section 2 presents the stability and convergence analysis of the single-variable integral-averaged interpolation operator. Section 3 provides the stability and convergence analysis of the n-variable integral-averaged interpolation operator. In Section 4, we present numerical examples to validate the corresponding theoretical results.