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A Stabilizer Free Weak Galerkin Finite Element Method for Elliptic Equation with Lower Regularity

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Abstract. This paper presents error analysis of a stabilizer free weak Galerkin finite element method (SFWG-FEM) for second-order elliptic equations with low regularity solutions. The standard error analysis of SFWG-FEM requires additional regularity on solutions, such as H^2-regularity for the second-order convergence. However, if the solutions are in H^{1+s} with $0 < s < 1$, numerical experiments show that the SFWG-FEM is also effective and stable with the $(1 + s)$-order convergence rate, so we develop a theoretical analysis for it. We introduce a standard H^2 finite element approximation for the elliptic problem, and then we apply the SFWG-FEM to approach this smooth approximating finite element solution. Finally, we establish the error analysis for SFWG-FEM with low regularity in both discrete H^1-norm and standard L^2-norm. The $(P_k(T), P_{k-1}(e), [P_{k+1}(T)]^d)$ elements with dimensions of space $d = 2, 3$ are employed and the numerical examples are tested to confirm the theory.

AMS subject classifications: 65N15, 65N30, 35J50
Key words: Stabilizer free weak Galerkin FEM, weak gradient, error estimate, lower regularity, second-order elliptic equation.

1. Introduction

The weak Galerkin finite element method (WG-FEM) is a useful numerical method for solving partial differential equations effectively. The WG-FEM is naturally derived from the standard finite element method (FEM) and the most important idea is to use the generalized functions and their weak derivatives which are defined as generalized distributions. The WG method is first introduced by Wang and Ye [17, 18] for the second-order elliptic equations, and a stabilizer term is added to WG-FEM in order to enforce the connection of discontinuous functions across element boundaries [10, 11]. Then the WG method finds applications in diverse areas including elliptic equations [9, 27], parabolic equations [30, 31, 34, 35], second-order linear wave

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equation [6], reaction-diffusion equations [8], Stokes equations [13, 16, 32], Maxwell equations [12, 14, 20], biharmonic equation [33], Cahn-Hilliard-Cook equation [5], stochastic parabolic equations [36, 37], eigenvalue problems [28, 29], and so on. Nevertheless, the stabilizer makes the finite element formulations and programming complex. To remove the stabilizer term, a stabilizer free weak Galerkin finite element method (SFWG-FEM) is introduced by Ye and Zhang [21] for the second-order elliptic equations on polytopal meshes. The main idea of SFWG-FEM is raising the degree of polynomials for weak gradient computation to increase the connectivity of weak functions. The SFWG-FEM is firstly proposed based on a traditional weak gradient definition, where the differential operator acts on the trial function, and then the element $(P_k(T), P_k(e), [P_j(T)]^d)$ has been demonstrated to be reliable if $j \geq k+n-1$, where n represents the number of element's edges/faces [22]. Subsequently, the requirement has been relaxed to be $j \geq k+n-2$ in general, especially to be $j \geq k+n-3$ when every edge ∂T is parallel to another one [3]. Various configurations of $(P_k(T), P_l(e), [P_j(T)]^d)$ with $l \geq k$ and $j \geq k+1$ lead to different schemes, cf. [1–3, 22–26]. Later on, the definition of weak gradient operator is modified by using the standard gradient of interior test functions instead of the divergence of trial functions [23]. And then, the finite element space for SFWG-FEM is released to be $(P_k(T), P_{k-1}(e), [P_{k+1}(T)]^d)$, which reduces the degree of freedom and maintains the same optimal order of convergence.

In order to reach the optimal convergence order for approximating the second-order elliptic equations, in many published literature on WG-FEM and SFWG-FEM, the solution is usually assumed to have at least H^2-smoothness [1, 23]. As a result, it is demonstrated that the convergence rates are at least $\mathcal{O}(h)$ in H^1-norm and $\mathcal{O}(h^2)$ in L^2-norm with mesh-size h, respectively. However, numerical experiments show that we have the $\mathcal{O}(h^s)$ convergence in $|\!|\!|\cdot|\!|\!|$-norm and $\mathcal{O}(h^{1+s})$ convergence in L^2-norm when the exact solution has only H^{1+s}-regularity $(0 < s < 1)$. The $(1+s)$-order L^2 convergence analysis on WG-FEM is accomplished in [19], and the core concern in [19] is only on analysis but without numerical experiments. There is no such theoretical analysis for SFWG-FEM, so in this paper, we are devoted to studying the convergence analysis on SFWG-FEM with low regularity, and to proving the discrete H^1-norm and L^2-norm convergence rates to be $\mathcal{O}(h^s)$ and $\mathcal{O}(h^{1+s})$, respectively. The numerical examples are also tested to confirm the theory. Our strategy is divided into two steps. Firstly, we use H^2-regular Argyris elements [4] as the bases of FEM to approximate the second-order elliptic equation whose solution has H^{1+s}-regularity, and then we get the optimal discrete H^1 and L^2 convergence orders to be of $\mathcal{O}(h^s)$ and $\mathcal{O}(h^{1+s})$, respectively. Secondly, we utilize the SFWG-FEM to approximate the H^2-regular finite element solution, and then we have the $\mathcal{O}(h)$ convergence in $|\!|\!|\cdot|\!|\!|$-norm and $\mathcal{O}(h^2)$ convergence in L^2-norm, respectively. As a consequence, we arrive at the optimal convergence rates to be $\mathcal{O}(h^s)$ in $|\!|\!|\cdot|\!|\!|$-norm and $\mathcal{O}(h^{1+s})$ in L^2-norm, respectively, if the exact solution of the second-order elliptic equation is only H^{1+s}-regular, which is an important supplementary for the new stabilizer free weak Galerkin finite element method theory.

The paper is organized as follows. In Section 2, we apply the Argyris finite element approximation to a second-order elliptic equation with H^{1+s}-regularity, and provide an