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Numerical Continuation and Bifurcation for Differential Geometric PDEs

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Abstract. We describe some differential geometric bifurcation problems and their treatment in the Matlab continuation and bifurcation toolbox `pde2path`. The continuation steps consist in solving the PDEs for the normal displacement of an immersed surface $X \subset \mathbb{R}^3$, with bifurcation detection and possible subsequent branch switching. The examples include minimal surfaces such as Enneper's surface and a Schwarz-P-family, some non-zero constant mean curvature surfaces such as liquid bridges, and some 4th order biomembrane models. In all of these we find interesting symmetry-breaking bifurcations. A few of these are (semi)analytically known and hence used as benchmarks.

AMS subject classifications: 35B32, 53C42, 65N30, 65P30

Key words: Numerical bifurcation, constant mean curvature, Helfrich functional, discrete differential geometry.

1. Introduction

Numerical continuation for partial differential equations (PDEs) yields the dependence of solutions on parameters, with special attention to bifurcation points, at which the local topological properties of the set of solutions change. These include fold points, where a branch folds back, and branch points, where different solution branches intersect. The list of toolboxes for numerical continuation of PDEs includes, e.g., `AUTO` [19] as a prototype package and role model, which in its standard setup for PDEs is mainly aimed at 1D boundary value problems, `Coco` [16], `BifurcationKit.jl` [78], and `pde2path` [73–75]. While all these packages in principle allow flexibility by delegating the PDE definition/discretization to the user, to the best of our knowledge they all

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rather focus on PDEs for functions $u : \Omega \times \Lambda \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^d$ is a fixed domain, $d = 1, 2$, or 3, $N \in \mathbb{N}$, and $\Lambda \subset \mathbb{R}^p$ is a parameter domain, or on PDEs for time-dependent functions $u : I \times \Omega \times \Lambda \to \mathbb{R}^N$, $I \subset \mathbb{R}$, which then includes the continuation and bifurcation of time periodic orbits.

However, differential geometric PDEs in parametric form may deal directly with manifolds, e.g., surfaces in 2D, which are not graphs over a fixed domain. There are well established numerical methods for the discretization of such PDEs, for instance the surface FEM [21], but there seem to be few algorithms or packages which combine these with continuation and bifurcation. Two notable exceptions are the algorithm from [13], and `SurfaceEvolver` [10], for which bifurcation aspects are for instance discussed in [11]. Here we present geometric PDE bifurcation problems from demos for the `Xcont` extension of `pde2path`. More implementation details of `Xcont` and the demos are presented in the tutorial [46], while here we focus on general principles and results, first for constant mean curvature surfaces, which are not necessarily graphs, and with the mean curvature, or the area or enclosed volume, as the primary bifurcation parameter, and second for some 4th order PDE biomembrane problems. See Fig. 1 for a preview of the type of solutions we compute.

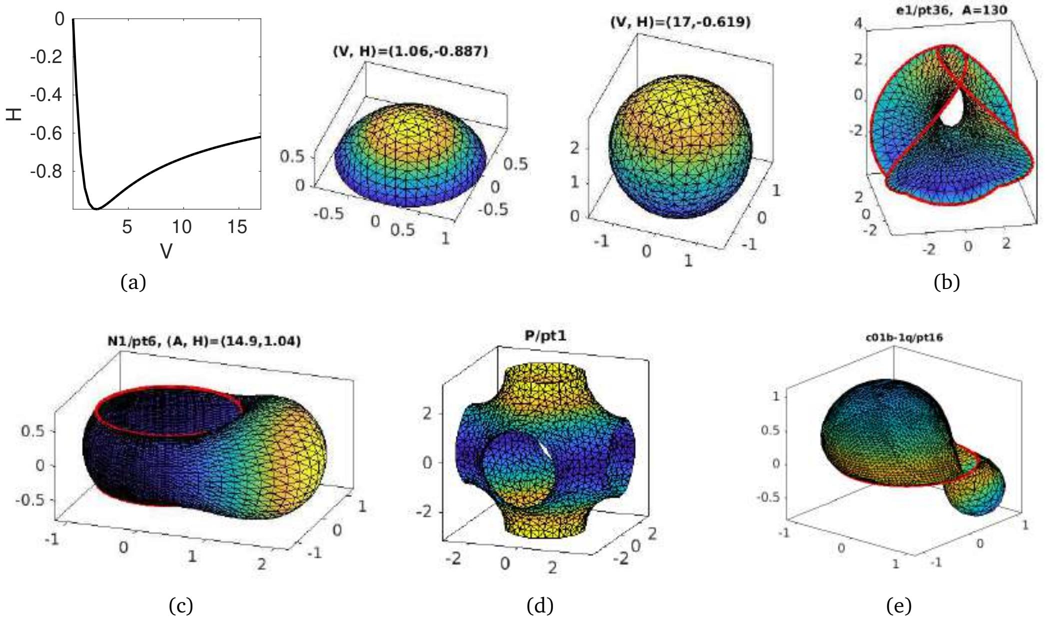

Figure 1: Preview of solutions (solution branches) we compute. (a) Mean curvature H (negative since we choose the outer normal N) over volume V for spherical caps, and sample solutions, Section 3.1. The colors indicate u in the last continuation step, yellow>blue, and thus besides giving visual structure to X indicate the direction of the continuation. (b) Enneper's minimal surface (a bounded part, with the boundary shown in red), Section 3.2. (c) A liquid bridge between two circles, with excess volume and hence after a symmetry breaking bifurcation, Section 3.3. (d) A Schwarz P surface, Section 3.4.1. (e) A Helfrich-type biomembrane cap after a symmetry breaking bifurcation. Samples (b)-(e) are each again from branches of solutions of the respective problems, see Figs. 6,7,10,16.