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An Extended Courant Element on a Polytope with Application in Approximating an Obstacle Problem

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Abstract. An extended Courant element is constructed on an n dimensional polytope K, which reduces to the usual Courant element when K is a simplex. The set of the degrees of freedom consists of function values at all vertices of K, while the shape function space P_K is formed by repeatedly using the harmonic extension from lower dimensional face to higher dimensional face. Several fundamental estimates are derived on this element under reasonable geometric assumptions. Moreover, the weak maximum principle holds for any function in P_K, which enables us to use the element for approximating an obstacle problem in three dimensions. The corresponding optimal error estimate in H^1-norm is also established. Numerical results are reported to verify theoretical findings.

AMS subject classifications: 65N30, 65N12, 65K15

Key words: Extended Courant element, virtual element, quasi-elliptic projection operators, obstacle problem.

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1. Introduction

The finite element method (FEM) is a type of handy and effective numerical methods for solving various industrial and engineering problems. Historically, the first finite element was proposed by Courant [16], which is now called the Courant element. In this case, a finite element function is a continuous piecewise linear function associated with a triangular mesh. However, only until the 1960s, Argyris, Clough, Zienkiewicz *et al.* re-discovered the element and used it to study structure analysis in engineering. The terminology finite element method was first raised in Clough's paper [15]. During the same period, the Chinese former mathematician Feng also proposed and analyzed the finite element method independently, which was named by

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him as the finite difference scheme based on variational principle (cf. [20]). We refer to [3, 10, 14, 34] and references therein for details about the comprehensive introduction of history, mathematical theories and applications of FEMs.

Following Ciarlet's convention (cf. [14]), a finite element is a triple $(K, \mathcal{P}_K, \mathcal{N}_K)$. Here, $K \subset \mathbb{R}^n$ (with n as a positive integer) is a bounded set with nonempty interior and piecewise smooth boundary, $\mathcal{P}_K$ is a finite-dimensional space of functions on K and $\mathcal{N}_K$ is a set of degrees of freedom (Dofs).

Let K be a triangle. The simplest choice of the Dofs $\mathcal{N}_K$ is the evaluation at all vertices of K. Then the Courant element can be represented by $(K, \mathbb{P}_1(K), \mathcal{N}_K)$. If K is a tetrahedron, the Courant element can be naturally extended to three dimensions as $(K, \mathbb{P}_1(K), \mathcal{N}_K)$ with $\mathcal{N}_K$ involving the evaluation at all vertices of K.

An interesting problem is how to extend the Courant element to a general polytope K in $\mathbb{R}^n$ with $n \geq 2$ a positive integer. In some sense, the most recently developed virtual element method (VEM) offered an answer to this issue (cf. [1, 4, 6, 9]). In fact, if K is a polygon, a finite element $(K, V_1(K), \mathcal{N}_K)$ was introduced in [4], where

$$V_1(K) = \big\{v \in H^1(K) : \Delta v = 0,\ v|_{\partial K} \in V_1(\partial K)\big\}, \tag{1.1}$$

$$V_1(\partial K) = \big\{v \in C(\partial K) : v|_e \in V_1(e) = \mathbb{P}_1(e),\ \forall\, e \subset \partial K\big\}, \tag{1.2}$$

while the set of Dofs consists of the function values at all vertices of K. If K is a triangle, this finite element is nothing but the Courant element. However, for a general polygon K, its shape function is implicitly defined, so this finite element is named as virtual element. The similar analogue is devised in three dimensions (cf. [1,28]), where the Dofs $\mathcal{N}_K$ also consist of function values at all vertices, and the corresponding shape function space $\overline{V}_1(K)$ is obtained by the harmonic extension from a boundary function belonging to a boundary space $W_1(\partial K)$ using the enhancement technique. Here,

$$\begin{aligned}
&\overline{V}_1(K) = \big\{v \in H^1(K) \cap C(\overline{K}) : \Delta v = 0,\ v|_{\partial K} \in W_1(\partial K)\big\},\\
&W_1(\partial K) = \big\{v \in C(\partial K) : v|_F \in W_1(F),\ \forall\, F \subset \partial K\big\},\\
&W_1(F) = \big\{v|_F \in \widetilde{V}_1(F) : (v, m_F)_F = \big(\Pi_F^\nabla v, m_F\big)_F,\ \forall\, m_F \in \mathbb{M}_1(F)\big\},\\
&\widetilde{V}_1(F) = \big\{v \in H^1(F) \cap C(\overline{F}) : \Delta_F v \in \mathbb{P}_1(F),\ v|_{\partial F} \in V_1(\partial F)\big\},
\end{aligned}$$

$\Pi_F^\nabla v$ is the standard elliptic projection, $\mathbb{M}_1(K)$ is the set of all scaled monomial on a domain K with degree up to 1 and $V_1(\partial F)$ is defined as in (1.2). In addition, applying the enhancement technique to the shape function space on K, one can get another virtual element $(K, W_1(K), \mathcal{N}_K)$ (cf. [1, 5, 8, 12, 21, 28]), where

$$\begin{aligned}
&W_1(K) = \big\{v \in \widetilde{V}_1(K) : (v, m_K)_K = \big(\Pi_K^\nabla v, m_K\big)_K,\ \forall\, m_K \in \mathbb{M}_1(K)\big\},\\
&\widetilde{V}_1(K) = \big\{v \in H^1(K) \cap C(\overline{K}) : \Delta v \in \mathbb{P}_1(K),\ v|_{\partial K} \in W_1(\partial K)\big\}.
\end{aligned}$$