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A Generalized Selectively Relaxed Matrix Splitting Preconditioning Strategy for Three-Dimensional Flux-Limited Multi-Group Radiation Diffusion Equations

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Abstract. Driven by the challenging task of pursuing the robust and accurate iterative numerical solution of the three-dimensional flux-limited multi-group radiation diffusion equations in an efficient and scalable manner, we propose and analyze a generalized matrix splitting preconditioning scheme with two selective relaxations and algebraic multigrid subsolves, introduce an algebraic quasi-optimal choice strategy to determine the involved parameters and consider its sequential implementation and two-level parallelization. A great deal of numerical results for typical unstructured twenty-group problems arising from realistic simulations of the hydrodynamic instability are presented and discussed to demonstrate the robustness, efficiency, strong and weak parallel scaling properties with up to 2,816 parallel processor cores together with the competitiveness of the proposed preconditioner when compared with several state-of-the-art monolithic and block preconditioning approaches.

AMS subject classifications: 65F08, 65N08, 65N55, 65Y05

Key words: Radiation diffusion equations, matrix splitting preconditioning, selective relaxation, algebraic multigrid, parallel and distributed computing.

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1. Introduction

The gray radiative transfer equations (GRTEs) are a type of simplified kinetic-scale mathematical model, which is commonly used to delineate the spatial-temporal evolu-

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tion of the radiative intensity and energy transfer of a radiation field with multicomponent background materials [16, 45]. This PDE system has multitudinous applications in inertial confinement fusions, optical remote sensings and massive star formations, which has spurred on a great number of highly accurate numerical researches because of the considerable importance but very high complexity. In the thick or high opacity regime, there must be rather intense interactions among photons of different frequencies, electrons and ions. In this circumstance, the photon mean free path approaches zero and the diffusive radiation behavior becomes dominant, whose simplest and most extensively applied approximation [26], in the mathematically less complicated and computationally cheaper yet numerically more accurate sense, is the second-order flux-limited nonlinear time-dependent multi-group radiation diffusion (MGD) equations on a spherically symmetrical bounded geometry

$$\begin{cases} \dfrac{\partial E_g}{\partial t} = \nabla\cdot\big(D_g(E_g)\nabla E_g\big) + c\big(\sigma_{Bg}B_g(T_E) - \sigma_{Pg}E_g\big) + S_g, \quad g=1,\dots,G, \\ \rho c_I\dfrac{\partial T_I}{\partial t} = \nabla\cdot\big(D_I(T_I)\nabla T_I\big) - w_{IE}(T_I - T_E), \\ \rho c_E\dfrac{\partial T_E}{\partial t} = \nabla\cdot\big(D_E(T_E)\nabla T_E\big) - c\displaystyle\sum_{g=1}^{G}\big(\sigma_{Bg}B_g(T_E) - \sigma_{Pg}E_g\big) + w_{IE}(T_I - T_E), \end{cases} \tag{1.1}$$

which search for the radiation energy density functions $E_1,\cdots,E_G$, the ion temperature function T_I and the electron temperature function T_E for some given density of medium ρ, specific heat capacities c_I and c_E, nonlinear thermal-conductivity coefficients $D_I(T_I)$ and $D_E(T_E)$ as well as the energy transfer coefficient w_{IE}, the g-th nonlinear radiation diffusion coefficient $D_g(E_g)$, the g-th scattering and absorption coefficients σ_{Bg} and σ_{Pg}, the g-th source item S_g and the g-th electron scattering energy density $B_g(T_E)$ for the photon frequency group index $g=1,\dots,G$.

Because photons are traveling with the speed of light, together with the high dimensionality as well as the strong coupled nonlinearity and discontinuity, the PDE system (1.1) would deserve an implicit temporal treatment to avoid an extremely small time step restriction which is inversely proportional to the speed of light when exploiting an explicit temporal discretization. The implicit temporal discretization which is often used in practice is the adaptive backward Eulerian scheme, which leads to a battery of semi-discrete nonlinear stationary PDE systems of the form

$$\begin{cases} \quad -\nabla\cdot\big(D_g(E_g)\nabla E_g\big) + \Big(\dfrac{1}{\Delta t_{k+1}} + c\sigma_{Pg}\Big)E_g - c\sigma_{Bg}B_g(T_E) \\ = S_g + \dfrac{1}{\Delta t_{k+1}}E_g^{(k)}, \quad g=1,\dots,G, \\ \quad -\nabla\cdot\big(D_I(T_I)\nabla T_I\big) + \Big(\dfrac{\rho c_I}{\Delta t_{k+1}} + w_{IE}\Big)T_I - w_{IE}T_E = \dfrac{\rho c_I}{\Delta t_{k+1}}T_I^{(k)}, \\ \quad -\nabla\cdot\big(D_E(T_E)\nabla T_E\big) + \Big(\dfrac{\rho c_E}{\Delta t_{k+1}} + w_{IE}\Big)T_E \\ \qquad +c\displaystyle\sum_{g=1}^{G}\sigma_{Bg}B_g(T_E) - c\sum_{g=1}^{G}\sigma_{Pg}E_g - w_{IE}T_I = \dfrac{\rho c_E}{\Delta t_{k+1}}T_E^{(k)} \end{cases} \tag{1.2}$$