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Splitting ADI Scheme for Fractional Laplacian Wave Equations

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Abstract. In this paper, we investigate the numerical solution of the two-dimensional fractional Laplacian wave equations. After splitting out the Riesz fractional derivatives from the fractional Laplacian, we treat the Riesz fractional derivatives with an implicit scheme while solving the rest part explicitly. Thanks to the tensor structure of the Riesz fractional derivatives, a splitting alternative direction implicit (S-ADI) scheme is proposed by incorporating an ADI remainder. Then the Gohberg-Semencul formula, combined with fast Fourier transform, is proposed to solve the derived Toeplitz linear systems at each time integration. Theoretically, we demonstrate that the S-ADI scheme is unconditionally stable and possesses second-order accuracy. Finally, numerical experiments are performed to demonstrate the accuracy and efficiency of the S-ADI scheme.

AMS subject classifications: 65F05, 65M06, 65M12, 65M15

Key words: Operator splitting, alternative direction implicit scheme, Gohberg-Semencul formula, fractional Laplacian wave equation.

1. Introduction

In this paper, we consider the numerical solution for the following initial boundary value problem of the fractional Laplacian wave equation:

$$\begin{cases} \partial_t^2 u(x,y,t) = -\kappa(-\Delta)^{\frac{\alpha}{2}} u(x,y,t) + g\big(u(x,y,t)\big), & (x,y)\in\Omega, \quad t\in(0,T], & (1.1)\\ u(x,y,t)=0, & (x,y)\in\Omega^c, \ \ t\in[0,T], & (1.2)\\ u(x,y,0)=\varphi_1(x,y), & (x,y)\in\Omega, & (1.3)\\ \partial_t u(x,y,0)=\varphi_2(x,y), & (x,y)\in\Omega, & (1.4)\end{cases}$$

where $\alpha\in(1,2)$ is the order of the fractional Laplacian, $T>0$ is the length of the time interval, $\Omega=(a,b)^2$ is a square domain in $\mathbb{R}^2$, $\Omega^c=\mathbb{R}^2\setminus\Omega$ and φ_1,φ_2 are two given

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functions on Ω. For a function $v:\mathbb{R}^d \mapsto \mathbb{R}$, the fractional Laplacian is defined by [9]

$$-(-\Delta)^{\frac{\alpha}{2}} v(\boldsymbol{x}) := -\frac{2^\alpha \Gamma((\alpha+d)/2)}{\pi^{\frac{d}{2}}|\Gamma(-\alpha/2)|} \text{ p.v. } \int_{\mathbb{R}^d} \frac{v(\boldsymbol{x})-v(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+\alpha}} \mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where 'p.v.' means that this integral takes its Cauchy principal value. When $d=1$, the fractional Laplacian is equivalent to the Riesz fractional derivative [1]

$$\partial_x^\alpha v(x) = -\frac{\partial_x^2 \int_{\mathbb{R}} |x-\xi|^{1-\alpha} v(\xi)\mathrm{d}\xi}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)}, \quad x \in \mathbb{R}.$$

The fractional Laplacian wave equation (1.1) with $\alpha = 2$ is the classical integer order wave equation, which has wide-ranging applications in quantum field theory [26], particle physics [7], and superconductor modelling [30]. In recent years, fractional calculus has garnered significant attention from researchers due to its successful applications in anomalous dispersion, random walk, and control systems [22, 32, 33]. As a result, the space fractional wave equation, which is a generalization of the integer order wave equation, has been extensively studied. Regarding solution theory, Karch [13] gave the long-time asymptotics for the linear damped fractional Laplacian wave equation. Chen *et al.* [2] presented a L^p estimate for the solution of the linear damped fractional Laplacian wave equation, which was then used to prove the existence of the global solution of the semilinear problem. Later, Ruan *et al.* [28] obtained the long-time decay estimate in Hardy spaces H^p for this linear problem. Fujiwara *et al.* [6] investigated the global well-posedness and the time-decay estimate for the solution of the damped fractional Laplacian wave equation with power nonlinearities. For the time-space fractional wave equation with spectral fractional Laplacian, Otárola and Salgado [23] proved the existence and uniqueness of the solution. Meanwhile, the regularity of the solution is derived. Djida *et al.* [4] obtained the existence and the L^p estimate of the classical solution of the semilinear time-space fractional wave equation. In the aspect of numerical solutions, for the one-dimensional (1D) fractional Laplacian wave equation, also known as the 1D Riesz fractional wave equation, numerous related works exist [5,19,20,38]. For the two-dimensional (2D) case, Wang and Shi [36] proposed an energy-conserving exponential scalar auxiliary variable spectral scheme for the fractional Laplacian wave equation on an unbounded domain. Hu *et al.* [11] employed a dissipation-preserving Crank-Nicolson pseudo-spectral method to solve the fractional Laplacian sine-Gordon equation with damping. Guo *et al.* [8] combined the Crank-Nicolson scheme with the exponential scalar auxiliary variable technique in the time direction and employed spectral-Galerkin method in the space direction to solve the coupled fractional Laplacian Klein-Gordon equation. However, the error estimations in those works are incomplete. Recently, based on the progress made in [9] about the finite difference approximation of the multi-dimensional fractional Laplacian operator, Hu *et al.* [10] proposed a dissipation-preserving difference scheme for the damped fractional Laplacian wave equation and established the unconditional stability and convergence of the proposed scheme.