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Fourier Convergence Analysis for a Fokker-Planck Equation of Tempered Fractional Langevin-Brownian Motion

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Abstract. Fourier stability analysis works well and is popular for the finite difference schemes of the linear partial differential equations. However, there are less works on the Fourier convergence analysis, and many of the existing ones require unreasonable assumptions. After removing the assumptions, we provide rigorous Fourier convergence analyses for the equation with one time fractional derivative in our previous work. In the current work, by using different ideas, we propose the rigorous Fourier convergence analyses for the equation with several time fractional derivatives, i.e., the Fokker-Planck equation of tempered fractional Langevin-Brownian motion, still without the strong assumptions. The numerical experiments are performed to confirm the theoretical results.

AMS subject classifications: 35R11, 65M06

Key words: Time-fractional Fokker-Planck model, $L1$-scheme, stability, Fourier convergence analysis.

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1. Introduction

Anomalous dynamics are ubiquitous in the nature world, especially in the complex system, the applications of which have a broad range, including physics [3], chemistry [19], and biology [38], etc. Tracing the history of stochastic dynamics, in 1827, Brown experimentally observed the irregular movement of pollen particles in water, being called Brownian motion (or normal diffusion), which dominated for quite some time. In fact, normal diffusion only represents a small part of the diffusion process. In the past decades, non-Brownian motion (or anomalous diffusion), which is widespread in nature, has been gradually discovered in experiments, covering the transport of charge carriers in amorphous semiconductors [15], the propagation of contaminants

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in groundwater [20], the movement of proteins in intracellular media [35], the transport of tracers in turbulent flows [34], the search patterns of foraging animals, and particle trajectories in dusty plasma [26], all of which cannot be described by Fick or Fourier's laws. In addition, numerous experiments have shown that the diffusion behavior of many systems in nature can generally be characterized by the mean squared displacement (MSD) of particles [27]. We usually distinguish between normal and anomalous diffusive processes according to the MSD (see [17, 39] and the references therein), i.e.,

$$\langle x^2(t)\rangle \sim t^{\beta},$$

where β is the anomalous diffusion index. We call it subdiffusion if $0<\beta<1$, normal diffusion if $\beta=1$, and superdiffusion if $\beta>1$. Especially, we call it underballistic hyperdiffusion, ballistic diffusion, and hyperballistic diffusion as $1<\beta<2$, $\beta=2$, and $\beta>2$, respectively, see, e.g., [29].

Anomalous diffusion is usually described by stochastic processes, including continuous time random walk model (CTRW), subordinate process, Lévy process, generalized Langevin equation [9–12, 22, 30], etc. In [7], the Klein-Kramers equation governing the probability density function of the Langevin equation with subordinate time scale transformation is derived, namely,

$$\begin{aligned}\frac{\partial u(x,v,t)}{\partial t} = &\frac{\overline{A}}{\beta}\Big[{}_0D_t^{2-2\beta}\, t-(1-\beta){}_0D_t^{1-2\beta}\Big]\frac{\partial^2 u(x,v,t)}{\partial x^2}\\ &- A\,{}_0D_t^{1-\beta}\frac{\partial u(x,v,t)}{\partial x}+\overline{A}\,{}_0D_t^{1-\beta}\frac{\partial^2 u(x,v,t)}{\partial x\partial v},\end{aligned} \tag{1.1}$$

where $0<\beta<1$, and the two constants $\overline{A}$ and A are given as

$$\begin{aligned}&\overline{A}=2\Gamma(2H)(2\lambda)^{-2H}A^2,\\ &A=\sqrt{k_BT}/\left[1+2\Gamma(2H)(2\lambda)^{-2H}\right]\end{aligned}$$

with $0<H<1$, $\lambda>0$, T being the absolute temperature of the environment, and k_B being the Boltzmann constant. Here, the mean square displacement of the particle described by Eq. (1.1) at time t is

$$\langle(\Delta X(t))^2\rangle \to \left(\frac{\sqrt{k_BT}A}{\beta\Gamma(2\beta)}-\frac{A^2}{(\beta\Gamma(\beta))^2}\right)t^{2\beta},$$

and, according to [32] the Riemann-Liouville fractional derivative ${}_0D_t^q$ is defined by

$${}_0D_t^q u(t)=\frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_0^t (t-s)^{m-1-q}u(s)\,ds,\quad m-1<q\le m,$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(z)=\int_0^{\infty} s^{z-1}e^{-s}ds,\quad \mathscr{R}(z)>0.$$