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Numerical Integration Formulas for Hypersingular Integrals

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Abstract. It is known that the solution of the Cauchy problem for partial differential equations of hyperbolic type can be reduced to singular integrals of a unique form. Laterly, singular integrals were called integrals in the sense of Hadamard or Hadamard integrals. In addition to equations of the hyperbolic type, Hadamard integrals are widely used in elasticity, electrodynamics, aerodynamics, and other vital areas of mechanics and mathematical physics. The exact evaluation of Hadamard integrals is possible only in exceptional cases, so that there is a need to develop approximate methods for their evaluation. In the present paper, we develop an optimal algorithm for the approximate calculation of Hadamard integrals. Here, we deal with finding the analytical form of the coefficients of an optimal quadrature formula. Numerical results show the validity and accuracy of the method.

AMS subject classifications: 65D30, 65D32

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1. Introduction

There are many problems, both in physics and technology and directly in various branches of mathematics, the solution of which requires calculating hypersingular integrals. Since direct computations of such integrals are possible only in exceptional cases, it becomes necessary to develop approximate methods. This paper is devoted to constructing an approximate method for calculating hypersingular integrals. Particular attention is paid to investigating the connection between methods for calculating singular and hypersingular integrals.

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Hypersingular integrals have several definitions. Here we adopt the following one.

Definition 1.1. *The Cauchy-Hadamard principal value of the integral*

$$Hg = \int_a^b \frac{g(x)}{(x-t)^p}\mathrm{d}x, \quad a < t < b, \quad p = 2, 3, \ldots \tag{1.1}$$

is defined as

$$\int_a^b \frac{g(x)}{(x-t)^p}\mathrm{d}x = \lim_{\varepsilon\to 0}\left\{\int_a^{t-\varepsilon} \frac{g(x)}{(x-t)^p}\mathrm{d}x + \int_{t+\varepsilon}^b \frac{g(x)}{(x-t)^p}\mathrm{d}x + \frac{\xi(\varepsilon)}{\varepsilon^{p-1}}\right\}, \tag{1.2}$$

where

$$\xi(\varepsilon) = -\sum_{k=0}^{p-2} \frac{g^{(k)}(t)}{k!} \cdot \frac{\varepsilon^k\left(1 + (-1)^{p-k}\right)}{p-k-1}$$

is a function constructed so that the limit exists [8, 10].

Theorem 1.1 ([16])**.** *The integral* (1.2) *exists for any function* $g(x) \in H_{p-1}(\alpha)$ *on the interval* $[a, b]$*, i.e.* $g^{(p-1)}(x) \in H(\alpha)$ *to the Hölder class of degree* α *on* $[a, b]$*.*

In the present work, we construct an optimal quadrature formula for the integral (1.1) in the case where $p = 2, 3, 4, \ldots$ and $a = 0, b = 1$, i.e., if

$$Hg = \int_0^1 \frac{g(x)}{(x-t)^p}\mathrm{d}x, \quad 0 < t < 1, \tag{1.3}$$

so that the kernel of the integral has a higher order singularity.

Recently, there has been increasing interest in Hadamard integrals of the form (1.1) and integral equations with integrals in the Hadamard sense, starting with the work of Hadamard and Schwartz. Several methods have been developed for the calculation of the hypersingular integrals. They are the discrete vortex method [5, 16], interpolation methods [3, 7], Gaussian methods [11, 12, 17, 23], the Newton-Cotes method [14, 15, 22], the transformation method [4, 6] and other numerical quadrature methods [13, 25]. In the discrete vortex method, a singular point is appropriate when it is midpoint of successive nodes. Many scientific papers are devoted to the construction of quadrature formulas with the interpolation methods. In these works, the nodes of the quadrature formula were taken as the roots of the Chebyshev polynomials of the first kind. In many physical applications, hypersingular integral equations need to be solved even when the function g is less smooth or unknown. In such cases, the Gaussian and transformation methods are efficient only if g is smooth enough. However, the accuracy of the numerical integration is significantly affected by the mesh selection. The use of the Gaussian method is limited by the mesh selection as the integrals are approximated at Gaussian points. The Newton-Cotes method is not suitable