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A Priori Error Estimate of Nonconforming Virtual Element Method for Convection Dominated Diffusion Equation

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Abstract. In this paper, a priori error estimate of the nonconforming virtual element method with Streamline Upwind/Petrov-Galerkin (SUPG) stabilization for convection dominated diffusion equation is discussed. The discrete scheme of the nonconforming virtual element method is constructed and a priori error estimate is derived with a norm containing jump term. Numerical experiments verify the a priori error estimate in a set of polygonal meshes.

AMS subject classifications: 65N30, 65N15, 76R99

Key words: Nonconforming virtual element method, SUPG stabilization, convection dominated diffusion equation, a priori error estimate.

1. Introduction

A priori error estimate of SUPG-nonconforming virtual element method (VEM) in the context for convection dominated diffusion equation is discussed. This kind of equation has many important applications, including river and air pollution, fluid flow and fluid heat conduction. Various stabilized schemes have been applied to solve this problem, for examples, SUPG methods [12, 22], edge stabilization methods [14, 23], continuous interior penalty (CIP) methods [13, 21] and so on. Recently, VEM has become

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one of the important methods to solve PDEs due to the extreme flexibility of mesh decomposition. From the initial VEM introduced in [2] to now, it has succeeded in solving different problems [6,15,17,18,24,26,32–35] and has derived mixed VEM [3,11,36], nonconforming VEM [1,25,27,37,39], stabilization-free VEM [4,9,16,30] and so on. The properties of nonconforming VEM are similar to the classical nonconforming FEM, such as the stability and weak continuity of shape function space. At the same time, compared with the conforming VEM, it is particularly suitable for solving high-order and high-dimensional problems, since they require much less degrees of freedom. The lowest-order nonconforming VEM degenerates into the well-known Crouzeix-Raviart element when the mesh is a triangular one [20].

Some works on the VEM for convection-diffusion-reaction are developed in the conforming case, where the discrete space is a subspace of the original one. In [7], the SUPG-VEM is studied for the convection-diffusion-reaction equation. In [5], a robust analysis of SUPG-VEM for convection dominated diffusion equation is discussed. By slightly modifying the SUPG format of [7] they propose a new way to discretize the convection term, which ultimately demonstrates the robustness of the parameters involved in the convergence estimation. In [29], they propose and analyze a local projection stabilization VEM for steady convection dominated diffusion equation and variational multiscale virtual element method for the convection dominated diffusion problem is considered in [38]. The work on the nonconforming VEM for advection-diffusion-reaction equation is found in [8]. When the mesh size is sufficiently small, the error estimate is obtained under the defined norm.

The model we consider is the convection dominaned reaction diffusion equation, and the coefficient of the diffusion term ε is a very small constant, for which the robustness estimation of the nonconforming virtual element method has not yet appeared. Firstly, a discrete scheme of the SUPG-nonconforming VEM is constructed. Like the classical nonconforming FEM [28], the jump terms are added to the discrete scheme. Because the shape functions can not be used explicitly in the VEM, the local projections are added to the jump terms in order to obtain a fully computable discrete scheme. Then a priori error estimate is derived with a norm containing the jump term. Finally numerical results show that the proposed method is in good agreement with the expected rate of convergence.

The outline of the paper is as follows. In the next section, the model problem is given. In Section 3, we introduce the nonconforming virtual element space and SUPG-nonconforming VEM formulation. In Section 4, some lemmas are introduced and a priori error estimate is derived. In Section 5, numerical examples are carried out to verify the theoretical analysis.

2. Model problem

First we introduce some notations. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with $\Gamma = \partial\Omega$. For an open bounded domain $\omega \subseteq \Omega$, we use the standard notation $\|\cdot\|_{W^{m,p}(\omega)}$ to denote the norm in the Sobolev space $W^{m,p}(\omega)$. For $p = 2$, we denote $H^m(\omega) :=$