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Error Estimates for the Time Discretization of a Semilinear Integrodifferential Parabolic Problem with Unknown Memory Kernel

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Abstract. This paper is devoted to the study of an inverse problem containing a semilinear integrodifferential parabolic equation with an unknown memory kernel. This equation is accompanied by a Robin boundary condition. The missing kernel can be recovered from an additional global measurement in integral form. In this contribution, an error analysis is performed for a time-discrete numerical scheme based on Backward Euler's Method. The theoretical results are supported by some numerical experiments.

AMS subject classifications: 47J35, 65M12, 65M32

Key words: Parabolic inverse problem, convolution kernel, error estimates.

1. Introduction

The aim of this paper is to derive error estimates for a time-discrete numerical scheme that approximates the solution of an inverse semilinear parabolic integrodifferential problem. This problem contains a Robin boundary condition and an unknown solely time-dependent memory kernel $K(t)$. More exactly, it is mathematically formulated as

$$\begin{cases} \partial_t u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) + K(t)h(\boldsymbol{x},t) - (K * \Delta u(\boldsymbol{x}))(t) = f(u(\boldsymbol{x},t)), & \\ & (\boldsymbol{x},t) \in \Omega \times (0,T], \\ \alpha(u(\boldsymbol{x},t)) + \nabla u(\boldsymbol{x},t) \cdot \boldsymbol{\nu} = g(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Gamma \times [0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

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with $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, a Lipschitz domain with boundary Γ and $[0,T]$, $T>0$, the time frame. The data functions h, f, g, α and u_0 are supposed to be known. The usual convolution in time is denoted by the symbol $*$, i.e.

$$(K * \Delta u(\boldsymbol{x}))(t) = \int_0^t K(t-s)\Delta u(\boldsymbol{x},s)\mathrm{d}s.$$

The convolution kernel $K(t)$ and the function $u(\boldsymbol{x},t)$ need to be reconstructed from the extra given measurement

$$\int_\Omega u(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = m(t), \qquad t \in [0,T]. \tag{1.2}$$

The identification of missing memory kernels in partial integrodifferential equations is relatively new in inverse problems. The first papers on this topic concern abstract parabolic and hyperbolic equations with memory [1–5]. These papers contain some local existence and global uniqueness results applying the contraction mapping principle. Other papers dealing with this topic are [6–14]. For instance, in [14], Colombo and Guidetti derived some local and global in time existence results for the recovery of solely time-dependent memory kernels in semilinear integrodifferential models. More specifically, they studied the evolution equation for materials with memory given by

$$\partial_t u = \Delta u + \int_0^t K(t-s)\Delta u(\boldsymbol{x},s)\,\mathrm{d}s + F(u), \quad \boldsymbol{x} \in \Omega_0 \subset \mathbb{R}^3, \quad t \in [0,T_0],$$

which corresponds with problem (1.1). More recent papers dealing with a similar problem setting are [15] and [16], in which the authors have used the global measurement (1.2) to reconstruct the kernel of a convolution of the form $K * u$ in a semilinear parabolic problem. Such types of integro-differential problems arise in the theory of reactive contaminant transport, cf. [17].

In [18], the development of a numerical algorithm for problems of type (1.1)-(1.2) has been provided under the condition that

$$\min_{t\in[0,T]} \left| \int_\Omega h(t) \right| \geqslant \omega > 0.$$

However, only weak convergence of the numerical approximations to the kernel K has been shown. The first goal of this paper is to slightly change the numerical scheme from [18], such that higher stability results can be obtained. These stability results are needed for the second goal, i.e., to perform an error analysis, from which the strong convergence of the numerical approximations to the kernel K follows. The last goal of this paper is to support the theoretical results with some numerical experiments. The acquired a priori estimates for the error estimates are complicated and deliver a possible solution approach for solving other integrodifferential problems.

The outline of this paper is as follows. First, the numerical scheme of [18] and some corresponding a priori estimates are adapted in Section 2. In the same section, also the