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Generalized Accelerated Hermitian and Skew-Hermitian Splitting Methods for Saddle-Point Problems

H. Noormohammadi Pour and H. Sadeghi Goughery *

Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran

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Abstract. We generalize the accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration methods for large sparse saddle-point problems. These methods involve four iteration parameters whose special choices can recover the preconditioned HSS and accelerated HSS iteration methods. Also a new efficient case is introduced and we theoretically prove that this new method converges to the unique solution of the saddle-point problem. Numerical experiments are used to further examine the effectiveness and robustness of iterations.

AMS subject classifications: 65F10; 65F15

Key words: saddle-point problem, Hermitian and skew-Hermitian splitting, preconditioning.

1. Introduction

We consider the iterative solution of large sparse saddle-point problems of the form

$$Ax = \begin{bmatrix} B & E \\ -E^* & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1.1}$$

where $B \in \mathbb{C}^{p\times p}$ is Hermitian positive definite, $0 \in \mathbb{C}^{q\times q}$ is zero, $E \in \mathbb{C}^{p\times q}$ has full column rank, $p \geq q$, $f \in \mathbb{C}^p$ and $g \in \mathbb{C}^q$. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1.1). Therefore, $A \in \mathbb{C}^{n\times n}$, with $n = p + q$, is a nonsingular, non-Hermitian, and positive semidefinite matrix. Linear systems of the form (1.1) arise in a variety of scientific and engineering applications, such as computational fluid dynamics, mixed finite element approximations of elliptic partial differential equations, constrained optimizations and constrained least-squares problems. For more detailed descriptions, see [1,2,7,8] and the references therein. Recent years, there are many effective iterative methods have

*Corresponding author. *Email address:* hsadeghi31@yahoo.com (H. Sadeghi Goughery)

been proposed by researchers, including splitting methods [2–4, 6, 7, 17–19], Uzawa-type schemes [9, 10, 13, 14, 20, 22, 24, 30, 31], and the preconditioned iterative methods [1, 7, 8, 11, 12, 21, 23, 25, 28]. Based on the Hermitian/skew-Hermitian (HS) splitting [15, 17, 29]

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*),$$

Bai *et al.* [5] proposed the Hermitian/skew-Hermitian splitting (HSS) iteration method. Benzi and Golub [11] discussed the convergence and the preconditioning properties of the Hermitian and skew-Hermitian splitting iteration method when it is used to solve the saddle-point problem (1.1) . Bai *et al.* [7] transformed the saddle-point problem (1.1) into an equivalent one, then applied the HSS method directly to the preconditioned block linear system, and established a class of preconditioned Hermitian/skew-Hermitian splitting (PHSS) iteration methods for the non-Hermitian positive semidefinite system of linear equations (1.1). In [2], Bai and Golub presented a class of accelerated Hermitian and skew-Hermitian splitting iteration methods (AHSS) for solving the large sparse saddle-point problem (1.1). These methods are two-parameter generalizations of the PHSS iteration methods studied in [7], and they can recover the PHSS methods as well as yield new ones by suitable choices of the two arbitrary parameters. In this paper we generalize accelerated HSS iteration methods for solving the saddle-point problem (1.1), and introduce an efficient case that is different from the PHSS and AHSS methods.

The paper is organized as follows. In Section 2, we review the PHSS and AHSS iteration methods and present generalized accelerated HSS iteration methods (GAHSS) for solving the saddle-point problem (1.1). In Section 3, we analyze the convergence properties of the new iteration method. In Section 4, numerical experiments are given to demonstrate the feasibility and effectiveness of the new iteration method. Finally, in Section 5, we draw some conclusions.

2. The GAHSS iteration

In this section, first we review the PHSS and AHSS iteration methods, see [2, 7] for more details, then present generalized accelerated HSS iteration methods (GAHSS) for solving the saddle-point problem (1.1). Consider matrices

$$P = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad \bar{E} = B^{-\frac{1}{2}} E C^{-\frac{1}{2}} \in \mathbb{C}^{p\times q}, \tag{2.1}$$

where $C \in \mathbb{C}^{q\times q}$ is a Hermitian positive definite submatrix, and define

$$\bar{A} = P^{-\frac{1}{2}} A P^{-\frac{1}{2}} = \begin{bmatrix} I_p & \bar{E} \\ -\bar{E}^* & 0 \end{bmatrix},$$